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Enhancements of triangulated categories

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Introduction

In this thesis we will study the theory of dg-enhancements of triangulated categories. In particular, we show the existence and uniqueness of dg enhancements for a wide class of triangulated categories. We will now give a brief introduction to these ideas.

The concept of localization is ubiquitous in mathematics; this can be roughly understood as the act of considering objects in a category up to a certain relation, that is in general weaker than that of isomorphism. There are two typical examples, one coming from algebra and one from topology. In algebra, when studying categories of chain complexes, one has two meaningful classes of such relations: homotopy equivalences and quasi-isomorphisms; localizing with respect to the first gives the homotopy category, while localizing with respect to the second gives the derived category. In topology one has a similar picture, where it can be interesting to consider objects up to homotopy equivalence or up to weak homotopy equivalence. Let us leave behind for now the topological example, and focus on the algebraic one; in particular, let us recall some facts about how the homotopy category is constructed. The key point here is that the basic relation is a notion of homotopy between morphisms: two morphisms are deemed equivalent if there exists a homotopy between them, and a morphism is considered a homotopy equivalence if it admits an inverse "up to homotopy". The homotopy category of chain complexes has fairly rich structure, captured by the notion of a triangulated category; this is roughly a category with a weak form of kernels and cokernels, and where kernels and cokernels coincide in a suitable sense. In particular, given a morphism in the category of chain complexes, one can define its cone: an object that, when considered up to homotopy, is a weak cokernel.

Here is where the problems begin: as a consequence of the axioms of a triangulated category, given any diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{f'} & B' \end{array}$$

that is commutative up to homotopy, there has to exist an induced morphism from the cone of f to the cone of f'. This exists, but its construction depends explicitly on the specific homotopy between $f'u$ and vf. In the homotopy category, where one has already forgotten everything except the fact that such a homotopy exists, it is not possible to define such a morphism in any canonical way. This issue is not peculiar to the homotopy category; one says (and after having appropriately defined its terms, proves) that in a triangulated category, the cone construction is not functorial. This creates some issues, in particular when trying to understand relationships between different triangulated categories. One is then led to consider *enhanced* categories: categories where, besides which morphisms are equivalent, one also records what are the homotopies, the homotopies between the homotopies, an so on. Of course, one would have to consider this enhanced category itself up to some sort of equivalence, as to not defy the initial intent of localizing. The formalization of the general concept of an enhanced category is not easy: Grothendieck, in the nineties, proposed the concept of a *dérivateur*, or derivator, while in the last years many authors have adopted the framework of ∞-categories, in several of their different incarnations.

Fortunately, if one restricts themselves to algebraic settings, the situation is noticeably simpler. After all, the homotopy category of chain complexes comes with a natural enhancement: given two chain complexes, one can easily define their *internal hom*: a chain complex encoding in degree 0 the morphisms between the objects, in degree -1 the homotopies between morphisms, and so on. This defines a *differential graded category* (from now on, dg-category), a category whose hom-spaces are in a natural way chain complexes; taking the homology of this dg-category (i.e. keeping the objects unchanged, but taking the homology of the hom-spaces), one recovers the homotopy category of chain complexes. Formally, one says that a dg-enhancement of a triangulated category is a dg-category whose homotopy category, defined as before by taking the homology of the hom-spaces, is equivalent to the first category. It turns out that most triangulated [1] categories appearing in algebra and algebraic geometry admit dg-enhancements; in particular, derived categories always admit dg-enhancements. Furthermore, dg-categories admit a quite natural form of equivalence, called quasi-equivalence. The main results presented in this thesis aim to show that this

[1] Enhancement of this type are mainly useful for triangulated categories, since taking the homology of a dg-category gives back a category that is always "almost" triangulated.

is a good point of view to take. First we will show that all "algebraic" triangulated categories admit a dg-enhancement, and then that for a very wide class of categories this enhancement is unique up to quasi-equivalence: the uniqueness result that we will prove is due to Lunts and Orlov, and was first proved in [LO10]. Moreover, and maybe more importantly, the notion of dg-category up to quasi equivalence is in general better behaved than that of triangulated category, not presenting any of the issues related to the non functoriality of the cone.[2]

[2] Besides having several other advantages, that we will not discuss here.

The thesis is structured in four chapters. The first one is dedicated to introducing the theory of localization and of triangulated categories, as well as their relationship in the form of Verdier and Bousfield localization; in the last section, we briefly introduce Quillen's model categories.

The second chapter is a short recollection of some known issues with triangulated categories. In particular, we prove there that it is almost never possible to define a functorial cone in a triangulated category, and that the category of morphisms of a triangulated category does not admit a triangulated structure.

Chapter three introduces the theory of dg-categories and dg-enhancements. All the basic definitions are given here, as well as the proofs of most foundational results. In the last part of the chapter, we give an overview of the homotopy theory of dg-categories and Drinfeld's construction of the dg-quotient.

In the fourth and last chapter we deal with the question of uniqueness of dg-enhancements. In particular, we show the proof of a results by Lunts and Orlov regarding the uniqueness of the enhancement for the derived category of a Grothendieck abelian category with a set of generators that are compact in the derived category. This proof is fairly long and takes up most of the chapter. In the final part, we discuss some recent developments that have followed the publication of [LO10].

Notations and conventions

We fix a universe $\mathbb{U}$. We will call small sets (or in general, sets) sets that are small with respect to $\mathbb{U}$.

We assume basic knowledge of category theory. By definition, all of our categories have small hom-set. We will use without distinction the words coproduct and direct sums, as well as the symbols $\coprod$ and $\bigoplus$ to denote the categorical coproduct; as a general rule, we will use $\bigoplus$ in additive settings and $\coprod$ in the general case. A small (co)limit will be a (co)limit indexed by a category whose class of objects is a small set. If $\mathcal{C}$ is a (eventually enriched) category, we will denote with

$$\mathrm{Hom}_{\mathcal{C}}(A, B)$$

the hom-space of morphisms between A and B, unless we want to stress the enriched nature of $\mathcal{C}$, in which case we will write

$$\mathcal{C}(A, B)$$

for the same object. In particular, when dealing with dg-categories, we will always use the second writing. All of our chain complexes will have cohomological notation, i.e. with differential increasing the index.

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Chapter 1

Preliminaries

1.1 Localizations, I

We begin with a brief introduction to the concept of localization of a category. References for this section are [GM02], [GZ67], [Wei94] and [Kra09].

1.1.1 Categories of fractions

Fix a category $\mathcal{C}$. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ makes a morphism s invertible if $F(s)$ is an isomorphism.

Definition 1.1. Let $\mathcal{S}$ be an arbitrary class of morphisms in $\mathcal{C}$. The localization of $\mathcal{C}$ at $\mathcal{S}$ is a category $\mathcal{C}[\mathcal{S}^{-1}]$ together with a functor

$$Q\colon \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}],$$

called quotient functor, satisfying the following properties:

- Q makes all the morphisms in $\mathcal{S}$ invertible;
- Any functor $G\colon C \to \mathcal{D}$ making all the morphisms in $\mathcal{S}$ invertible factors uniquely through Q via a functor $\tilde{\mathcal{G}}\colon \mathcal{C}[\mathcal{S}^{-1}] \to \mathcal{D}$.

From the definition it follows immediately that if a localization exists, it is unique. We can also prove that, modulo a set-theoretic issue, localizations always exist. Set $\mathrm{Ob}(\mathcal{C}[\mathcal{S}^{-1}]) = \mathrm{Ob}(\mathcal{C})$. Then, for two objects A, B define $\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B)$ in the following way: first consider the class of all the sequences

$$A \to A' \leftarrow A'' \to \cdots \to B' \leftarrow B$$

where all the right-pointing arrows lie in $\mathcal{S}$. Beware that we do not require the sequence to be an alternating sequence of right-pointing and left-pointing arrows, since we do allow multiple arrows in the same direction; also, the first and last arrow can be either outward or inward. Now quotient this class by the following relations:

- Two consecutive morphism pointing in the same direction can be replaced by their composition;
- For any morphism $A \xrightarrow{s} B$ in $\mathcal{S}$, the sequence

$$A \xrightarrow{s} B \xleftarrow{s} A$$

  can be replaced with

$$A \xrightarrow{\mathrm{id}} A.$$

Define $\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B)$ as this quotient; composition is given by concatenation of sequences, while the identity is given the sequence

$$A \xrightarrow{\mathrm{id}} A.$$

Finally, define the functor Q as the identity on objects and sending a morphism $f\colon A \to B$ to the sequence of length one

$$A \xrightarrow{f} B.$$

It is straightforward to verify that the just defined object satisfies the universal property of a localization. However in general, for fixed objects A and B, $\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(A, B)$ does not form a small set, not making $\mathcal{C}[\mathcal{S}^{-1}]$ actually a category; furthermore, it is virtually impossible to give useful descriptions of the morphism spaces between two objects, with zigzags arbitrary length not being prone to explicit calculation. In practice though, most of the times one is not interested in inverting arbitrary families of morphisms, but classes of morphisms arising in some natural way, that are usually easier to deal with: we now briefly recall two different approaches to localizing a category in some specialized cases. The first, the calculus of fractions, is in principle much less refined than the second, Quillen's theory of model categories; it is nonetheless important to discuss both of them, since the calculus of fractions as a crucial application in defining Verdier quotients.

1.1.2 Multiplicative systems

A class of morphisms $\mathcal{S}$ is said to admit a calculus of left fractions if the following properties are satisfied:

(LF1) All the identities lie in $\mathcal{S}$, and compositions of morphisms in $\mathcal{S}$ lie in $\mathcal{S}$;

(LF2) Each pair of morphisms $A \xleftarrow{s} X \xrightarrow{f} B$ with $s \in \mathcal{S}$ can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ A & \xrightarrow[f']{} & X' \end{array}$$

with $s' \in \mathcal{S}$;

(LF3) For any diagram of the form

$$X' \xrightarrow{s} X \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

such that $s \in \mathcal{S}$ and $\alpha \circ s = \beta \circ s$, there exists a morphism $t \colon Y \to Y'$ such that $t \in \mathcal{S}$ and $t \circ \alpha = t \circ \beta$.

If the dual conditions are satisfied, $\mathcal{S}$ is said to admit a calculus of right fractions. If $\mathcal{S}$ admits a calculus of left fractions and a calculus of right fractions, it is called a multiplicative system.

If $\mathcal{S}$ admits a calculus of left fractions, it is possible to give a very explicit description of the hom-spaces of $\mathcal{C}[\mathcal{S}^{-1}]$. Given two objects X and Y, any morphism between them in $\mathcal{C}[\mathcal{S}^{-1}]$ is represented by a "roof" or left fraction

$$X \xrightarrow{f} Z \xleftarrow{s} Y$$

with $s \in \mathcal{S}$; such a roof should be interpreted as the formal fraction $s^{-1}f$. It is a useful exercise to interpret in this light the meaning of condition LF2 : it tells us that we can "move the denominator from the left to the right", in analogy the the localization of a non commutative ring. Two roofs

$$X \xrightarrow{f} Z \xleftarrow{s} Y \qquad \text{and} \qquad X \xrightarrow{g} Z' \xleftarrow{t} Y$$

are considered equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \overset{f}{\nearrow} & \downarrow & \overset{s}{\nwarrow} & \\ X & \xrightarrow{h} & Z'' & \xleftarrow{l} & Y \\ & \underset{g}{\searrow} & \uparrow & \underset{t}{\swarrow} & \\ & & Z' & & \end{array}$$

with $l \in \mathcal{S}$. The composition of two roofs

$$\begin{array}{ccccccccc} & & Z & & & & Z' & & \\ & \overset{f}{\nearrow} & & \overset{s}{\nwarrow} & & \overset{g}{\nearrow} & & \overset{t}{\nwarrow} & \\ X & & & & Y & & & & W \end{array}$$

is defined by applying condition LF2 to the middle "V", and finding the diagram

$$\begin{array}{ccccccccc} & & & & Z'' & & & & \\ & & & \overset{t}{\nearrow} & & \overset{s'}{\nwarrow} & & & \\ & & Z & & & & Z' & & \\ & \overset{f}{\nearrow} & & \overset{s}{\nwarrow} & & \overset{g}{\nearrow} & & \overset{t}{\nwarrow} & \\ X & & & & Y & & & & W. \end{array}$$

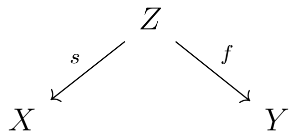
Since $s' \in \mathcal{S}$ and $\mathcal{S}$ is closed under compositions, we can define the compositions as the roof

$$\begin{array}{ccccc} & & Z'' & & \\ & \overset{tf}{\nearrow} & & \overset{s't}{\nwarrow} & \\ X & & & & W. \end{array}$$

This (see for example [GM02] or [Kra09]) defines a category, and the so constructed category satisfies the universal property of the localization. The functor $Q\colon \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ is the identity on objects and sends a morphism $X \xrightarrow{f} Y$ to the roof

$$\begin{array}{ccccc} & & Y & & \\ & \overset{f}{\nearrow} & & \overset{\mathrm{id}_y}{\nwarrow} & \\ X & & & & Y. \end{array}$$

Dually, if $\mathcal{S}$ admits a calculus of right fractions, we can define a right fraction as a roof



with $s \in \mathcal{S}$, and proceed in the same way as before.

Remark. We have seen that, in the presence of a multiplicative system, the hom-spaces of a localization admit an explicit and simple construction. Nonetheless, this does not guarantee the smallness of the hom-spaces, since the vertex of the roof is an arbitrary object in the category; in fact, localized categories with non-small hom-spaces arise in nature - that is, provided a very loose definition of nature - see for example [Kra09, Example 4.15].

It is often useful to find either a left or a right adjoint to the quotient functor, since in that case we are able to realize the hom-spaces in the quotient as hom-spaces between particular objects in the non-localized category: indeed, suppose that Q has a right adjoint ρ. Then, for any $X, Y \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}[\mathcal{S}^{-1}]}(QX, QY) \cong \mathrm{Hom}_{\mathcal{C}}(X, \rho QY).$$

It turns out that it is fairly easy to find conditions for the quotient functor to admit adjoints, just using the fact that $\mathcal{S}$ is a multiplicative system; this is done in full generality in [Kra09]. Since we will only be interested in the case of triangulated categories, we will not deal with the general case, delaying this discussion to Chapter 1.3.

1.2 Triangulated categories

1.2.1 Some basic homological algebra

In this section, $\mathbf{A}$ will represent a fixed abelian category. We now briefly review some definitions and basic results about chain complexes in $\mathbf{A}$; good sources for this are [Wei94] and [GM02]. Recall that a (cohomological) chain complex A of objects in $\mathbf{A}$ is a collection $\{A^i\}_{i\in\mathbb{Z}}$ of objects in $\mathbf{A}$ together with morphisms $d_i \colon A^i \to A^{i+1}$ such that $d_{i+1} \circ d_i = 0$ for every i. We will often suppress the subscript i from the morphism d_i, writing the above condition simply as $d^2 = 0$. We will denote with $Z^iA \subseteq A^i$ the kernel of d_i, and with $B^iA \subseteq A^i$ the image of d_{i-1}. We will call the elements of Z^iA the i-cycles (or closed elements), while the elements of B^iA will be called i-boundaries. Since the condition $d^2 = 0$ implies that $B^iA \subseteq Z^iA$, we can define the i-th homology of a chain complex A as $H^iA = Z^iA/B^iA$. A chain map $f\colon A \to B$ is a collection $f^i\colon A^i \to B^i$ of morphisms in $\mathbf{A}$ commuting with the differentials, i.e. such that $d_i f^i = f^{i+1} d_i$. By definition, a chain map f sends cycles to cycles and boundaries to boundaries, so induces a map in homology $f^i_*\colon H^i A \to H^i B$. If f^i_* is an isomorphism for every i, f is said to be a quasi-isomorphism. Given a chain complex A, we can define the shifted complex $A[n]$ by setting $A[n]^i = A^{n+i}$ and $d_{A[n]} = (-1)^n d_A$. a chain complex A is said to be bounded above if $A^n = 0$ for $n >> 0$; bounded below if $A^n = 0$ for $n << 0$ and bounded if it is both bounded above and bounded below. If $H^i A = 0$ for every i, A is said to be acyclic.

Definition 1.2. The category $\mathbf{C}(\mathbf{A})$ of chain complexes of objects in $\mathbf{A}$ is the category whose objects are chain complexes of objects in $\mathbf{A}$ and whose morphisms are chain maps.

$\mathbf{C}(\mathbf{A})$ is an abelian category, with kernels and cokernels being computed degree-wise. Recall now that a homotopy h between two chain maps $f, g\colon A \to B$ is a collection of morphisms $h^i\colon A^i \to B^{i-1}$ such that $d_{i-1}h^i + h^{i+1}d_i = f^i - g^i$ for each i. Two chain maps are said to be homotopic if there exists a homotopy between them; in that case we will write $f \sim^h g$[1]. A nullhomotopy of a chain map is an homotopy between it and the zero morphism. It is straightforward to verify that being homotopic is an equivalence relation, and that two homotopic maps induce the same morphism in homology. A chain map $f\colon A \to B$ is said to be a homotopy equivalence if there exists a chain map $g\colon B \to A$ such that $fg \sim^h \mathrm{id}_B$ and $gf \sim^h \mathrm{id}_A$; since homotopic morphisms induce the same morphism in homology, an homotopy equivalence is in particular a quasi-isomorphism. Note that the converse is in general not true.

[1]The h here stands for the word homotopy, not for the specific homotopy between f and g.

In this case, we have two classes of morphisms that we may be interested in inverting: the homotopy equivalences and the quasi-isomorphisms. Unfortunately, as it is, neither of them form a multiplicative system. However, unlike quasi-isomorphisms, homotopy equivalences are already defined having a built-in “almost-inverse”: to turn them into isomorphisms, it should be enough to consider them as honest inverses, that is to identify homotopic morphisms as equivalent.

Definition 1.3. The homotopy category of chain complexes $\mathcal{K}(\mathbf{A})$ is defined as having the same objects as $\mathbf{C}(\mathbf{A})$, and hom-spaces

$$\mathrm{Hom}_{\mathcal{K}(\mathbf{A})}(A, B) = \mathrm{Hom}_{\mathbf{C}(\mathbf{A})}(A, B) \big/ \sim^h,$$

where $\sim^h$ represents the homotopy relation.

Sometimes $\mathcal{K}(\mathbf{A})$ is just called the homotopy category. It is easy to verify that this definition is well posed, i.e. that composition with homotopic morphisms yields homotopic morphisms and that $\mathrm{Hom}_{\mathcal{K}(\mathbf{A})}(A, B)$ inherits the additive structure from $\mathrm{Hom}_{\mathbf{C}(\mathbf{A})}(A, B)$. We also have an additive functor $Q\colon \mathbf{C}(\mathbf{A}) \to \mathcal{K}(\mathbf{A})$ acting as the identity on objects and sending a morphism to its equivalence class in the quotient. The following proposition is then very natural.

Proposition 1.4. *$\mathcal{K}(\mathbf{A})$, equipped with the functor Q is the localization of $\mathbf{C}(\mathbf{A})$ at the homotopy equivalences.*

Sketch of the proof. We want to prove that any functor $F\colon \mathbf{C}(\mathbf{A}) \to \mathcal{C}$ to an arbitrary category inverting all homotopy equivalences factors through $\mathcal{K}(\mathbf{A})$. This will follow if we prove that if $f \sim^h g$, then $F(f) = F(g)$. In order to do this, consider the following construction: for a given chain complex A, define the chain complex $\mathrm{Cyl}(A)$ by setting

$$\mathrm{Cyl}(A)^n = A^n \oplus A^{n+1} \oplus A^n$$

and

$$d(x^n, y^{n+1}, z^n) = (dx^n + y^{n+1}, -dy^{n+1}, dz^n - y^{n+1}).$$

It is a straightforward exercise to prove that $\mathrm{Cyl}(A)$ is indeed a chain complex and that chain maps $\varphi\colon \mathrm{Cyl}(A) \to B$ correspond to triples (f, h, g) where $f, g\colon A \to B$ are chain maps and h is a homotopy between f and g. This bijection is made explicit by defining the two natural inclusions (in the first and third summand) $i_0, i_1\colon A \to \mathrm{Cyl}(A)$ so that, for any $\varphi\colon \mathrm{Cyl}(A) \to B$, we find $f = \varphi \circ i_0$ and $g = \varphi \circ i_1$. Furthermore, i_0 and i_1 are homotopy equivalences: their (common) homotopy inverse is given by the morphism $p\colon \mathrm{Cyl}(A) \to A$ representing the trivial homotopy between the identity of A and itself. We are now ready to conclude the proof: let $f, g\colon A \to B$ be two homotopic morphisms, and let $\varphi\colon \mathrm{Cyl}(A) \to B$ be the corresponding morphism. Since F inverts all homotopy equivalences, we know that $F(i_0) = F(i_1) = F(p)^{-1}$. Since $\varphi \circ i_0 = f$ and $\varphi \circ i_1 = g$, it follows that $F(f) = F(g) = F(\varphi) \circ F(p)^{-1}$. □

The advantage of this approach is two-fold: to begin with, we now have a very explicit description of the localized category. Moreover, it now happens that the class of quasi-isomorphisms, that is well defined in $\mathcal{K}(\mathbf{A})$ in virtue of the fact that homotopic chain maps induce the same morphism in homology, is now a multiplicative system.

Definition 1.5. The derived category $D(\mathbf{A})$ of $\mathbf{A}$ is defined as the localization of $\mathcal{K}(\mathbf{A})$ at the quasi-isomorphisms.

It is true (but not completely obvious) that we could also have defined abstractly $D(\mathbf{A})$ as the localization of $\mathbf{C}(\mathbf{A})$ at the quasi-isomorphisms obtaining the same object, although in a way that is much harder to describe. Since $D(\mathbf{A})$ is maybe better understood as a Verdier quotient, we will give more details on this construction and the fact that quasi-isomorphisms form a multiplicative system in the following sections: for now, we will keep focusing on $\mathcal{K}(\mathbf{A})$. For future uses, we also define the full subcategories $\mathcal{K}^+(\mathbf{A})$, $\mathcal{K}^-(\mathbf{A})$ and $\mathcal{K}^b(\mathbf{A})$ of $\mathcal{K}(\mathbf{A})$ composed by respectively bounded below, bounded above and bounded chain complexes.

1.2.2 Triangulated categories

This section contains barely any proofs, aiming mainly at defining the objects and discussing their main properties. Nonetheless, we try to give full references for the stated results. A comprehensive source on triangulated categories is Neeman's book [Nee01b].

We have already seen that the category $\mathbf{C}(\mathbf{A})$ is an abelian category, and that its additive structure on the hom-spaces descends to $\mathcal{K}(\mathbf{A})$. It is also easy to see that it admits finite biproducts, and that the canonical functor $Q\colon \mathbf{C}(\mathbf{A}) \to \mathcal{K}(\mathbf{A})$ preserves both products and coproducts. However, $\mathcal{K}(\mathbf{A})$ is almost never itself abelian, since, for example, in general $\mathcal{K}(\mathbf{A})$ does not possess cokernels: cokernels in $\mathbf{C}(\mathbf{A})$ are not cokernels in $\mathcal{K}(\mathbf{A})$. The best approximation of a cokernel which exists in $\mathcal{K}(\mathbf{A})$ the cone of a morphism.

Definition 1.6. Let A, B be chain complexes and $f\colon A \to B$ a chain map. Its cone $C(f)$ is defined as the chain complex having in degree n the object

$$C(f)^n = A^{n+1} \oplus B^n$$

and differential

$$d(x^{n+1}, y^n) = (-d_A x^{n+1}, d_B y^n + f^{n+1}(x^{n+1}))$$

for $x^{n+1} \in A^{n+1}$ and $y^n \in B^n$.

It is easy to verify that $C(f)$ is a chain complex and that there exist natural chain maps $i\colon B \to C(f)$ and $p\colon C(f) \to A[1]$ assembling into a short exact sequence

$$0 \to B \to C(f) \to A[1] \to 0.$$

As a consequence (via the long exact sequence in homology), a morphism is a quasi-isomorphism if and only if its cone is acyclic. One can also prove that a morphism is a homotopy equivalence if and only if its cone is isomorphic to 0 in $\mathcal{K}(\mathbf{A})$.

Foreshadowing remark. $C(f)$ is a homotopy cokernel of f, in the following sense: morphisms from $C(f)$ to D correspond to morphisms $g\colon B \to D$ together with a nullhomotopy h of $g \circ f$. It should be stressed that those are not the same as morphisms $g\colon B \to D$ for which there exists a nullhomotopy of the composition, as that would just be a cokernel in $\mathcal{K}(\mathbf{A})$; the specific nullhomotopy is part of the datum of the morphism from the cone. This, as we will see in the following chapters, is problematic: the homotopy category $\mathcal{K}(\mathbf{A})$ knows when two morphisms are homotopic, but not what the homotopy between them is. If a construction necessitates the explicit homotopy (versus the sole knowledge that such a homotopy exists) this construction is not possible in $\mathcal{K}(\mathbf{A})$. This fact will lead us to consider "enhanced" categories, i.e. categories where one records, besides which morphisms are homotopic, also what are the homotopies (and what are the homotopies between the homotopies...) □

So $\mathcal{K}(\mathbf{A})$ is not abelian, but nonetheless has much more structure than that of a general additive category. It turns out that a good formalization of its properties is that of a triangulated category. Let us now state its definition.

Definition 1.7. Let $\mathcal{T}$ be an additive category, equipped with an additive automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$. We will write $X[n]$ for $\Sigma^n X$, and will call the functor Σ the shift functor. A candidate triangle in $\mathcal{T}$ is a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

A morphism of candidate triangles is a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

It is said to be an isomorphism if u, v, w are isomorphisms.

Definition 1.8. Let $\mathcal{T}$ be an additive category. A structure of triangulated category on $\mathcal{T}$ is the datum of an additive automorphism $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and of a class of candidate triangles, called distinguished triangles satisfying the following axioms:

TR1 a) The candidate triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$ is distinguished for any X.

b) Every triangle isomorphic to a distinguished triangle is itself distinguished.

c) Every morphism $X \xrightarrow{f} Y$ can be completed to a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$. We will cone Z the cone of f.

TR2 A candidate triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished if and only the "rotated" candidate triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.

TR3 Every diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

where the rows are distinguished triangles can be completed (not necessarily in a unique way) to a morphism of triangles by a morphism $Z \xrightarrow{w} Z'$.

TR4 Suppose we have three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

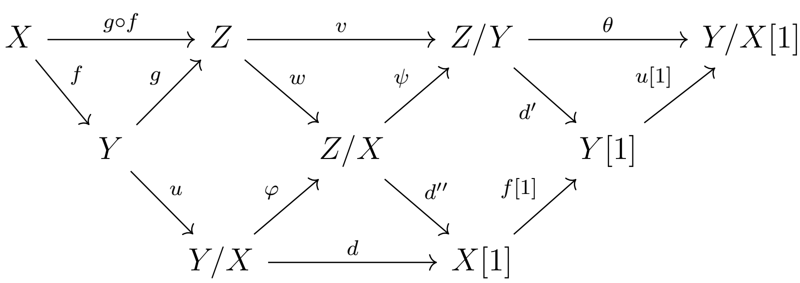
$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1].$$

Then there exists a fourth distinguished triangle

$$Y/X \xrightarrow{\varphi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

making the diagram



commute. Beware that with Y/X we are not indicating the actual quotient of Y by X (that would not make sense, being $\mathcal{T}$ not necessarily abelian), but just any object completing the morphism $X \xrightarrow{f} Y$ to a distinguished triangle.

A triangulated category is a category together with the choice of a triangulated structure.

Remark. Axiom $TR4$ (known as the Verdier axiom, or the octahedral axiom) has several different forms present in the literature, all equivalent to each other; The one shown here is the one present in [Lur17], while [GM02] and [Wei94] have use a different form of it (namely, the one giving the axiom its "octahedral" name). We do not give here a detailed discussion of it, referring instead to [Nee01b] and [May66]. Nonetheless, let us record some facts: because of its complicated form, many have wondered whether the octahedral axiom is in fact an axiom, or can be in any way derived from axioms TR1-TR3. To this day the question is open, as there does not exist either a proof of the fact that the first three axioms imply the fourth, or an example of a category where TR1-TR3 hold but TR4 does not. This can be seen as another (very mundane) advantage of enhanced categories over triangulated categories: in all the theories of enhanced categories, TR4 is a theorem following from the usually simpler axioms of an enhanced category.

In [May66], the author proves that axiom TR3, the axiom that most explicitly encodes the non functoriality of the cone, can in fact be deduced from axiom TR4; We could have then given the definition of a triangulated category only listing axioms TR1, TR2 and TR4. Nonetheless, since TR3 is more useful in basic computations, most authors keep it as an axiom.

Remark. If $\mathcal{T}$ has the structure of a triangulated category, then $\mathcal{T}^{op}$ has a natural triangulated structure induced by that of $\mathcal{T}$: its shift is given by (the opposite of) the inverse to Σ, while the distinguished triangles are those of the form

$$Z \xleftarrow{f} Y \xleftarrow{g} X \xleftarrow{h} Z[-1]$$

such that the triangle

$$X \xrightarrow{g} Y \xrightarrow{h} Z \xrightarrow{-f[1]} X[1]$$

is distinguished.

Remark. It is noted in [Nee01b] that it is not necessary to require for $\mathcal{T}$ be additive, but it is enough for it to be preadditive (i.e. such that its hom-spaces have a natural additive structure) and for a zero object to exist: axioms T1-T4 then imply that finite biproducts always exist.

Let's see what this definition means in the promised example of $\mathcal{T} = \mathcal{K}(\mathbf{A})$. In this case, we take the endofunctor Σ to be the shift of complexes; it is trivial to verify that this is well-defined at the homotopy level. We call standard triangles those of the form

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} X[1]$$

for some morphism $X \xrightarrow{f} Y$ in $\mathbf{C}(\mathbf{A})$, and define the distinguished triangles to be those isomorphic to a standard triangle. We then have

Theorem 1.9. *The category $\mathcal{K}(\mathbf{A})$ has a natural structure of triangulated category, with the shift functor given by the shift of complexes and distinguished triangles as above. Similarly, the categories $\mathcal{K}^+(\mathbf{A})$, $\mathcal{K}^-(\mathbf{A})$ and $\mathcal{K}^b(\mathbf{A})$ of $\mathcal{K}(\mathbf{A})$ all carry a natural triangulated structure.*

We do not prove this here, instead referring to [GM02, Theorem IV.1.9] or [Wei94, Proposition 10.2.4]. The reader can find a sketch of the proof of an equivalent statement in section 3.4.

Remark. The definition of standard triangle as was given might seem ambiguous; since f is a morphism in $\mathcal{K}(\mathbf{A})$, to take its cone we have to choose a representative in $\mathbf{C}(\mathbf{A})$. A first fix is to say that for any morphism in $\mathcal{K}(\mathbf{A})$ we define a set of standard triangles, one for each representative. Over the course of the proof we will nonetheless prove that homotopic morphisms have homotopy equivalent cones; however as usual, the isomorphism between the cones of homotopic morphisms will depend on the explicit homotopy.

Let us now state a couple of useful definitions.

Definition 1.10. Let $\mathcal{T}, \mathcal{S}$ be triangulated categories. An additive functor $F \colon \mathcal{T} \to \mathcal{S}$ is said to be exact if there exists a natural isomorphism $F\Sigma \cong \Sigma F$ (where the first sigma represents the shift in $\mathcal{T}$ and the second that of $\mathcal{S}$) and for every distinguished triangle in $\mathcal{T}$

$$X \to Y \to Z \to X[1]$$

the induced candidate triangle in $\mathcal{S}$

$$FX \to FY \to FZ \to FX[1]$$

is distinguished.

Definition 1.11. Let $\mathcal{A}$ be an abelian category. An additive functor $F \colon \mathcal{T} \to \mathcal{A}$ is called homological if for any distinguished triangle in $\mathcal{T}$

$$X \to Y \to Z \to X[1]$$

the sequence

$$F(X) \to F(Y) \to F(Z)$$

is exact. An additive functor $G\colon \mathcal{T}^{op} \to \mathcal{A}$ is called cohomological if for any distinguished triangle in $\mathcal{T}$

$$X \to Y \to Z \to X[1]$$

the induced sequence

$$G(Z) \to G(Y) \to G(X)$$

is exact.

By axiom TR2, if $F\colon \mathcal{T}^{op} \to \mathcal{A}$ is cohomological and

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle, then the whole sequence

$$\cdots \to F(Z)[-1] \to F(X) \to F(Y) \to F(Z) \to F(X)[1] \to \cdots$$

is exact.

Example 1.1. Let $\mathbf{A}$ be an abelian category and let Ab be the category of abelian groups. Consider the functor

$$H^0\colon \mathbf{C}(\mathbf{A}) \to \mathrm{Ab}$$

assigning to a chain complex its homology in degree 0. It sends all homotopy equivalences to isomorphisms, so it defines a functor

$$H^0\colon \mathcal{K}(\mathbf{A}) \to \mathrm{Ab}\,.$$

By definition, we have that $H^0(X[i]) = H^i X$. Since every triangle is isomorphic to a standard triangle and standard triangles come from the exact sequence

$$0 \to Y \xrightarrow{i} C(f) \xrightarrow{p} X[1] \to 0,$$

the long exact sequence in homology gives that the functor H^0 (and in general also the functor $H^n = H^0 \circ \Sigma^n$) is homological .

Definition 1.12. A full subcategory $\mathcal{S} \subseteq \mathcal{T}$ of a triangulated category is called a triangulated subcategory if the following conditions hold:

- $\mathcal{S}$ is closed under isomorphisms;
- $\mathcal{S}$ is closed under shifts: if an object X lies in $\mathcal{S}$, so do its shifts $X[n]$.

- $\mathcal{S}$ is closed under taking cones: if $X \to Y \to Z \to X[1]$ is a distinguished triangle and $X, Y \in \mathcal{S}$, so does Z.

We will say that a triangulated subcategory is thick if it is closed under direct summands. If $\mathcal{T}$ has small coproducts, we will say that a triangulated subcategory is localizing if it is thick and closed under small coproducts. Note that from the third condition it follows that (as long as $\mathcal{S}$ is non-empty), the zero object always lies in $\mathcal{S}$. Similarly, the first condition is actually redundant, since the shift of any object X can be recovered up to isomorphism as the cone of the unique morphism $X \to 0$.

Remark. It can be proved ([Nee01b, Proposition 1.6.8]) that if a triangulated subcategory admits countable coproducts, it is automatically thick. So in particular, we could remove the thickness hypothesis from the definition of localizing subcategory.

We now state some basic results about abstract triangulated categories.

Proposition 1.13. *Let X be an object in a triangulated category $\mathcal{T}$. Then the functors*

$$\mathrm{Hom}_{\mathcal{T}}(X, -) \colon \mathcal{T} \to \mathrm{Ab} \quad \textit{and} \quad \mathrm{Hom}_{\mathcal{T}}(-, X) \colon \mathcal{T}^{op} \to \mathrm{Ab}$$

are respectively homological and cohomological.

Proof. [GM02, Proposition IV.1.3]. □

In particular, if

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle, the composition of any two consecutive arrow is always the zero morphism. Moreover, a morphism is an isomorphism if and only if its cone is isomorphic to 0.

Corollary 1.14. *Suppose we have a morphism of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

with both rows distinguished. If u and v are isomorphisms, so is w.

As a consequence of this fact, the object completing a given morphism f to a distinguished triangle (as per axiom TR1) is unique up to isomorphism. Indeed, suppose that we have two distinguished triangles

$$X \xrightarrow{f} Y \to C \to X[1]$$

and

$$X \xrightarrow{f} Y \to C' \to X[1]$$

this gives immediately a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & C & \longrightarrow & X[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow{f} & Y & \longrightarrow & C' & \longrightarrow & X[1] \end{array}$$

that by axiom TR3 can be completed to a morphism of triangles via a morphism $C \xrightarrow{h} C'$. By the corollary above, h is an isomorphism. Therefore, given a morphism in a triangulated category we can talk about its cone without ambiguity. When doing this however, we should be careful in remembering that the isomorphism between two different cones is not unique, and we will see in Chapter 2 that this non uniqueness is not fixable (for example by using the axiom of choice) in the triangulated setting.

Proposition 1.15. *Let $F\colon \mathcal{T} \to \mathcal{S}$ be an exact functor between triangulated categories. If F admits a left or right adjoint G, then G is automatically exact. In particular, inverses to exact functors are exact.*

Proof. [Sta, Lemma 13.7.1.]. □

1.2.3 Compact generation

Definition 1.16. Let $\mathcal{T}$ be a triangulated category. An object $X \in \mathcal{T}$ is said to be compact if for every set of objects $\{Y_i\}$ such that their coproduct exists in $\mathcal{T}$ the natural morphism

$$\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \to \mathrm{Hom}_{\mathcal{T}}(X, \bigoplus_i Y_i)$$

is an isomorphism. We will denote with $\mathcal{T}^c$ the class of all compact objects in a triangulated category.

This is equivalent to saying that any morphism from X to a coproduct of an arbitrary set of objects factors through a coproduct of a finite number of those. Compactness is a very strong "smallness" condition: it can be interpreted as a triangulated analogue to being finite-dimensional.

Definition 1.17. Let $\mathcal{S} \subseteq \mathcal{T}^c$ be a class of compact objects in a triangulated category. Suppose also that $\mathcal{T}$ admits small coproducts. We say that $\mathcal{S}$ compactly generates $\mathcal{T}$ if the following condition holds: for any object $X \in \mathcal{T}$,

$$X = 0 \iff \mathrm{Hom}_{\mathcal{T}}(S[n], X) = 0 \text{ for any } S \in \mathcal{S}, n \in \mathbb{Z}.$$

In this case we call $\mathcal{S}$ a set of compact generators for $\mathcal{T}$.

Using the fact that $\mathcal{T}$ admits all coproducts, it can be proved ([SS03, Lemma 2.2.1]) that the condition that $\mathcal{S} \subseteq \mathcal{T}^c$ generates $\mathcal{T}$ is equivalent to the fact that $\mathcal{T}$ coincides with the smallest localizing subcategory containing $\mathcal{S}$. Note that it is also true (see [Nee01b, Proposition 1.16]) that both products and coproducts in any triangulated category commute with shifts, and also ([Nee01b, Proposition 1.2.1]) that products and coproducts of distinguished triangles are again distinguished.

If a triangulated category admits a (small) set of compact generators, we say that it is compactly generated.

Lemma 1.18. *Let $\mathcal{T}$ be triangulated category, and suppose that it is generated by a set $\mathcal{R} \subseteq \mathcal{T}^c$. Let $X \xrightarrow{f} Y$ be a morphism in $\mathcal{T}$. Suppose that for any $R \in \mathcal{R}$, f induces a bijection*

$$\mathrm{Hom}_{\mathcal{T}}(R, X) \cong \mathrm{Hom}_{\mathcal{T}}(R, Y).$$

Then f is an isomorphism.

Proof. Consider the full subcategory $\mathcal{S} \subseteq \mathcal{C}$ of objects S such that the map

$$\mathrm{Hom}_{\mathcal{T}}(S, X) \to \mathrm{Hom}_{\mathcal{T}}(S, Y).$$

is an isomorphism. By hypothesis it contains $\mathcal{R}$. If we prove that $\mathcal{S}$ is localizing, we obtain that $\mathcal{S} = \mathcal{T}$ and are then done by Yoneda's lemma. It is obvious that $\mathcal{S}$ is closed under isomorphisms. First we prove that $\mathcal{S}$ is closed under arbitrary coproducts: if $\{S_i\} \subseteq \mathcal{S}$, we have

$$\mathrm{Hom}_{\mathcal{T}}\big(\bigoplus_i S_i, X\big) \cong \prod_i \mathrm{Hom}_T(S_i, X) \cong \prod_i \mathrm{Hom}_T(S_i, Y) \cong \mathrm{Hom}_{\mathcal{T}}\big(\bigoplus_i S_i, Y\big),$$

so $\bigoplus_i S_i \in \mathcal{S}$. Similarly, $\mathcal{S}$ is closed under taking cones: take a morphism $g\colon S_1 \to S_2$ between objects of $\mathcal{S}$ and let

$$S_1 \xrightarrow{g} S_2 \to P \to S_1$$

be a distinguished triangle. We then have, writing Hom instead of $\mathrm{Hom}_{\mathcal{T}}$, a commutative diagram

$$\begin{array}{ccccccccc} \mathrm{Hom}(S_2[1], X) & \to & \mathrm{Hom}(S_1[1], X) & \to & \mathrm{Hom}(P, X) & \to & \mathrm{Hom}(S_2, X) & \to & \mathrm{Hom}(S_1, X) \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ \mathrm{Hom}(S_2[1],)Y & \to & \mathrm{Hom}(S_1[1], Y) & \to & \mathrm{Hom}(P, Y) & \to & \mathrm{Hom}(S_2, Y) & \to & \mathrm{Hom}(S_1, Y) \end{array}$$

where the rows are exact and all vertical arrows except the middle one are isomorphisms. By the five lemma, the middle arrow is an isomorphism and we are done. □

Lemma 1.19. *Let $\mathcal{T}$ be a triangulated category generated by a set $\mathcal{R} \subseteq \mathcal{T}^c$. Let $\mathcal{S}$ be any triangulated category, and*

$$F\colon \mathcal{T} \to \mathcal{S}$$

an exact functor, and suppose that the following hold:

- *FX is compact in $\mathcal{S}$ for all $X \in \mathcal{R}$*
- *F is fully faithful when restricted to $\mathcal{R}$, i.e.*

  $$\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{S}}(FX, FY)$$

  is an isomorphism for all $X, Y \in \mathcal{R}$.

Then F is fully faithful.

Proof. Consider the full subcategory $\mathcal{A} \subseteq \mathcal{T}$ of all objects A such that

$$\mathrm{Hom}_{\mathcal{T}}(X, A) \to \mathrm{Hom}_{\mathcal{S}}(FX, FA)$$

is an isomorphism for all $X \in \mathcal{R}$. By hypothesis, $\mathcal{R} \subseteq \mathcal{A}$. $\mathcal{A}$ is a triangulated subcategory: it is obviously closed under isomorphism, and if $A \xrightarrow{f} B$ is a morphism between objects of $\mathcal{A}$, we have a triangle

$$A \xrightarrow{f} B \to C \to A[1],$$

where C is the cone of C. Therefore, reasoning as in the Lemma above, $C \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is localizing: For any set $\{A_i\} \subseteq \mathcal{A}$, we have

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{T}}(X, \bigoplus_i A_i) \cong \bigoplus_i \mathrm{Hom}_{\mathcal{T}}(X, A_i) \cong \bigoplus_i \mathrm{Hom}_{\mathcal{S}}(FX, FA_i) \cong \\ &\cong \mathrm{Hom}_{\mathcal{S}}(FX, \bigoplus FA_i),\end{aligned}$$

so $\bigoplus_i A_i \in \mathcal{A}$. Note that we have crucially used the fact that X and FX are compact. Therefore, $\mathcal{A} = \mathcal{T}$. Now consider the full subcategory $\mathcal{B} \subseteq \mathcal{T}$ of objects B such that

$$\mathrm{Hom}_{\mathcal{T}}(B, X) \to \mathrm{Hom}_{\mathcal{S}}(FB, FX)$$

is an isomorphism for all $X \in \mathcal{T}$. We have just proved that $\mathcal{R} \subseteq \mathcal{B}$, and repeating the argument above we can prove that $\mathcal{R}$ is a localizing subcategory. Note now that this time we do not need the assumption that X and FX be compact, but only the universal property of a coproduct. Then $\mathcal{B} = \mathcal{T}$, and the claim is proved. □

Note that both Lemma 1.18 and Lemma 1.19 continue to hold even if $\mathcal{R}$ is not a set.

The following is a fundamental result in the study of triangulated categories, and is called Brown's representability theorem for (compactly generated) triangulated categories.

Theorem 1.20. *Let $\mathcal{T}$ be a compactly generated triangulated category, and suppose that it admits small coproducts. Then, an exact functor*

$$F\colon \mathcal{T} \to \mathcal{S}$$

admits a right adjoint if and only if it preserves small coproducts.

Proof. [Nee01b, Theorem 8.4.4] is a slightly generalized form of this theorem. □

1.2.4 Homotopy colimits

Let $\mathcal{T}$ be a triangulated category and

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} X_3 \to \cdots$$

a sequence of morphisms in $\mathcal{T}$. Suppose that the coproduct $\bigoplus_i X_i$ exists in $\mathcal{T}$. Define a map J[2]

$$\begin{aligned}J\colon \bigoplus_i X_i &\to \bigoplus_i X_i\\ (x_0, x_1, x_2, \ldots) &\to (x_0, x_1 - i_0(x_0), x_2 - i_1(x_1), \ldots)\end{aligned}$$

[2] Often the functor J is denoted as $\mathrm{id} - \mathrm{shift}$.

Definition 1.21. The homotopy colimit of the sequence is defined as any object $\underrightarrow{\mathrm{hocolim}}_i X_i$ completing J to an exact triangle

$$\bigoplus_i X_i \xrightarrow{J} \bigoplus_i X_i \to \underrightarrow{\mathrm{hocolim}}_i X_i \to \bigoplus_i X_i[1].$$

Note that $\underrightarrow{\mathrm{hocolim}}_i X_i$ is only defined up to a non unique isomorphism.

Lemma 1.22. *Let $\mathcal{T}$ be a triangulated category that admits small coproducts, and let*

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \to \cdots$$

be a sequence of morphisms in $\mathcal{T}$. Suppose that $X \in \mathcal{T}$ is a compact object. Then there is a natural isomorphism

$$\underrightarrow{\mathrm{colim}}_i \operatorname{Hom}_{\mathcal{T}}(X, X_i) \to \operatorname{Hom}_{\mathcal{T}}(X, \underrightarrow{\mathrm{hocolim}}_i X_i).$$

In other words, the functor $\operatorname{Hom}_{\mathcal{T}}(X, -)$ carries homotopy colimits to colimits of abelian groups.

Proof. [Nee92b, Lemma 1.5]. □

Example 1.2. If $\mathcal{T} = \mathcal{K}(\mathbf{A})$, and if a representative for each i_n is chosen, one can show that there exists a quasi-isomorphism

$$\underrightarrow{\mathrm{hocolim}}_i X_i \to \underrightarrow{\mathrm{colim}} X_i.$$

1.3 Localizations, II

In this section we deal with the theory of localizations for triangulated categories. We will see that in this case those are usually easier to understand.

Let now $\mathcal{T}$ be a triangulated category. Recall that a class $\mathcal{S} \subseteq \operatorname{Mor}(\mathcal{T})$ is called a multiplicative system if it admits both a calculus of left and right fractions. We say that a multiplicative system is compatible with the triangulation if the following two conditions are satisfied:

- For every morphism $\sigma \in \mathcal{S}$, $\sigma[i] \in \mathcal{S}$ for any $i \in \mathbb{Z}$.
- For any morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\ \downarrow u & & \downarrow v & & \downarrow w & & \downarrow u[1] \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & X'[1] \end{array}$$

with $u, v \in \mathcal{S}$, there exists a morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & T & \xrightarrow{h} & X[1] \\ {\scriptstyle u}\downarrow & & {\scriptstyle v}\downarrow & & {\scriptstyle w'}\downarrow & & \downarrow{\scriptstyle u[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & T' & \xrightarrow{h'} & X'[1] \end{array}$$

with $w' \in \mathcal{S}$.

Proposition 1.23. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subseteq \mathrm{Mor}(\mathcal{T})$ a multiplicative system compatible with the triangulation. Assume further that $\mathcal{T}[\mathcal{S}^{-1}]$ has small hom-sets. Then $\mathcal{T}[\mathcal{S}^{-1}]$ admits a unique triangulated structure making the quotient functor $Q\colon \mathcal{T} \to \mathcal{T}[\mathcal{S}^{-1}]$ exact.*

Proof. [Ver96, Théorème II.2.2.6]. □

Multiplicative systems of this type arise naturally in two (related) ways. We begin with the first.

Proposition 1.24. *Let $F\colon \mathcal{T} \to \mathbf{A}$ be a homological (or dually, cohomological) functor. Then the class of morphisms σ such that $F(\sigma[n])$ is an isomorphism for any n is a multiplicative system compatible with the triangulation.*

Proof. [Kra09, Lemma 4.4.2]. □

As a consequence of this proposition, the class of quasi-isomorphism forms a multiplicative system compatible with the triangulation of $\mathcal{K}(\mathbf{A})$.

Another source of multiplicative systems comes from triangulated subcategories.

1.3.1 Verdier localization

To a triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ we can associate the class $\mathcal{FS} \subseteq \mathrm{Mor}(\mathcal{T})$ composed of all morphisms $X \xrightarrow{f} Y$ such that the cone of f lies in $\mathcal{S}$.

Proposition 1.25. *For any triangulate subcategory $\mathcal{S} \subseteq \mathcal{T}$, the class $\mathcal{FS}$ is a multiplicative system compatible with the triangulation of $\mathcal{T}$.*

Proof. [Kra09, Lemma 4.6.1]. □

It is useful to remember that the class of distinguished triangles in the quotient is by definition the class of all triangles isomorphisms to images via the quotient of distinguished triangles in $\mathcal{T}$.

Definition 1.26. The Verdier quotient ${}^{\mathcal{T}}\!/_{\mathcal{S}}$ is by definition the localization

$$ {}^{\mathcal{T}}\!/_{\mathcal{S}} = \mathcal{T}[\mathcal{F}\mathcal{S}^{-1}] $$

equipped with the quotient functor $Q\colon \mathcal{T} \to {}^{\mathcal{T}}\!/_{\mathcal{S}}$.

By Proposition 1.23, the quotient ${}^{\mathcal{T}}\!/_{\mathcal{S}}$ carries a unique triangulated structure making Q exact. Note also that the quotient acts as the identity on objects. We say that a functor $F\colon \mathcal{T} \to \mathcal{T}'$ between triangulated (or just additive) categories annihilates a full subcategory $\mathcal{S} \subseteq \mathcal{T}$ if it sends every object of $\mathcal{S}$ to an object isomorphic to 0. Since the cone of the morphism $0 \to X$ is always isomorphic to X, the quotient $\mathcal{T} \to {}^{\mathcal{T}}\!/_{\mathcal{S}}$ annihilates $\mathcal{S}$. We have the following characterization;

Proposition 1.27. *Let $\mathcal{T}$ be a triangulated category, and $\mathcal{S} \subseteq \mathcal{T}$ a triangulated subcategory. Any exact functor $F\colon \mathcal{T} \to \mathcal{T}'$ annihilating $\mathcal{S}$ factors uniquely through Q via an exact functor ${}^{\mathcal{T}}\!/_{\mathcal{S}} \to \mathcal{T}'$.*

Proof. This is proved in [Kra09, Proposition 4.6.2] using the fact that a functor annihilating $\mathcal{S}$ necessarily inverts all the morphisms in $\mathcal{F}\mathcal{S}$. □

Remark. Sometimes, Verdier quotients are only considered when $\mathcal{S}$ is a thick subcategory. The reason for this is that when $\mathcal{S}$ is thick, then it coincides with the kernel of the quotient, i.e. the subcategory of objects annihilated by the quotient; if $\mathcal{S}$ is not thick, the kernel of the quotient is the closure of $\mathcal{S}$ under direct summands.

Proposition 1.28. *Suppose that $\mathcal{T}$ admits small coproducts, and that $\mathcal{S}$ is a localizing subcategory. Then the quotient ${}^{\mathcal{T}}\!/_{\mathcal{S}}$ admits small coproducts and the quotient functor $\mathcal{T} \to {}^{\mathcal{T}}\!/_{\mathcal{S}}$ preserves coproducts.*

Proof. [Nee01b, Lemma 3.2.10]. □

Example 1.3. If $\mathbf{A}$ is an abelian category, we can define its derived category $D(\mathbf{A})$ as the localization of $\mathcal{K}(\mathbf{A})$ at the quasi-isomorphism or, equivalently, as the Verdier quotient of $\mathcal{K}(\mathbf{A})$ by the acyclic complexes: since the cone of a morphism is acyclic if and only if the morphism is a quasi-isomorphism, these definitions coincide. We can also define the bounded below, bounded above and bounded derived categories $D^+(\mathbf{A})$, $D^-(\mathbf{A})$ and $D^b(\mathbf{A})$ by taking the localization of $\mathcal{K}^+(\mathbf{A})$, $\mathcal{K}^-(\mathbf{A})$ and $\mathcal{K}^b(\mathbf{A})$ at the quasi-isomorphisms. A technical advantage of the derived category over the homotopy category is that it is possible to prove (see Lemma 3.63) that triangles in the derived category coincide precisely with short exact sequences. This is not the case in the homotopy category: all triangles are by definition isomorphic to the short exact sequence

$$0 \to B \xrightarrow{i} C(f) \xrightarrow{p} A[1] \to 0$$

for some morphism $A \xrightarrow{f} B$, but not all exact sequences induce exact triangles in the homotopy category.

Often, it is useful to consider the relationship between localizations and subcategories. Consider a triangulated category $\mathcal{T}$ with two triangulated subcategories $\mathcal{T}'$ and $\mathcal{S}$. Define $\mathcal{S}' = \mathcal{S} \cap \mathcal{T}'$, and we have by definition that $\mathcal{F}_{\mathcal{T}'}\mathcal{S}' = \mathcal{F}_{\mathcal{T}}(\mathcal{S}) \cap \mathcal{S}'$[3]. The composition

[3] $\mathcal{F}_{\mathcal{T}}\mathcal{S}$ indicates the class of morphisms in $\mathcal{T}$ whose cone lies in $\mathcal{S}$.

$$\mathcal{T}' \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

annihilates $\mathcal{S}'$ and therefore induces a functor

$$\mathcal{T}'/\mathcal{S}' \to \mathcal{T}/\mathcal{S}.$$

We then have the following proposition.

Proposition 1.29. *Suppose that one of the following conditions hold:*

- *Any morphism from an object of $\mathcal{S}$ to an object of $\mathcal{T}'$ factors through an object of $\mathcal{S}'$;*
- *Any morphism from an object of $\mathcal{T}'$ to an object of $\mathcal{S}$ factors through an object of $\mathcal{S}'$.*

Then the natural functor

$$\mathcal{T}'/\mathcal{S}' \to \mathcal{T}/\mathcal{S}.$$

is fully faithful.

Proof. [Kra09, Lemma 4.7.1]. □

Definition 1.30. Take $\mathcal{T}$, $\mathcal{S}$ as above. We then define the two full subcategories

$$\mathcal{S}^\perp = \{Y \in \mathcal{T} | \operatorname{Hom}_{\mathcal{T}}(X, Y) = 0 \quad \forall X \in \mathcal{S}\}$$

and

$$^\perp\mathcal{S} = \{X \in \mathcal{T} | \operatorname{Hom}_{\mathcal{T}}(X, Y) = 0 \quad \forall Y \in \mathcal{S}\}.$$

We call $\mathcal{S}^\perp$ and $^\perp\mathcal{S}$ orthogonal subcategories.

We have the following useful characterization:

Proposition 1.31. *Let $Y \in \mathcal{T}$. The following are equivalent:*

- $Y \in \mathcal{S}^{\perp}$*;*
- *The quotient functor induces a bijection*

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$$

  for any $X \in \mathcal{T}$*.*

Dually, for an an element $X \in \mathcal{T}$ the following are equivalent:

- $X \in {}^{\perp}\mathcal{S}$*;*
- *The quotient functor induces a bijection*

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \cong \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$$

  for any $Y \in \mathcal{T}$*.*

Proof. [Kra09, Lemma 4.8.1] □

1.3.2 Bousfield localization

We are now ready to study the existence of adjoints of the quotient functor.

Proposition 1.32. *Let $\mathcal{T}$ be a triangulated category, and $\mathcal{S}$ a thick subcategory. The following are equivalent:*

- *The quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a right adjoint;*
- *For each $X \in \mathcal{T}$ there exists a distinguished triangle*

$$X' \to X \to X'' \to X'[1]$$

  with X' in $\mathcal{S}$ and X'' in $\mathcal{S}^{\perp}$.

Dually, the following are equivalent:

- *The quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a left adjoint;*
- *For each $X \in \mathcal{T}$ there exists a distinguished triangle*

$$X' \to X \to X'' \to X'[1]$$

  with X' in ${}^{\perp}\mathcal{S}$ and X'' in $\mathcal{S}$.

Furthermore when they exist, both the left and right adjoint to the quotient are fully faithful.

Proof. We refer to [Kra09, Proposition 4.9.1] for the details, but we give a description of the adjoint functor: let's consider the first case. To construct the adjoint, we fix for each $X \in {}^{\mathcal{T}}\!/_{\mathcal{S}}$ an exact triangle in $\mathcal{T}$

$$X' \to X \to X'' \to X[1]$$

as above. Define then

$$\mu \colon {}^{\mathcal{T}}\!/_{\mathcal{S}} \to \mathcal{T}$$

by setting $\mu(X) = X''$. This might appear strange, since we have been hammering the reader with warnings about how constructions involving triangles are not canonical. However, the specific form of this triangle (namely, the fact that X' lies in $\mathcal{S}$), makes the construction work. Indeed, since $X' \in \mathcal{S}$, the morphism $X \to X''$ is an isomorphism in ${}^{\mathcal{T}}\!/_{\mathcal{S}}$. Therefore, using Proposition 1.31 we have natural bijections

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X'', Y'') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{T}}(\mu(X), \mu(Y))$$

which show ρ to be a fully faithful functor. Finally, again Proposition 1.31 proves that ρ is a right adjoint to the quotient. In the case of the left adjoint, one can set $\lambda(X) = X'$ with X' as in the second triangle. □

When the quotient functor $\mathcal{T} \to {}^{\mathcal{T}}\!/_{\mathcal{S}}$ admits a right adjoint, it is called a Bousfield localization; when it admits a left adjoint, a Bousfield colocalization.

We conclude this section with a different way to find (right) adjoints to the quotient functor, via Brown's representability theorem.

Remark. In the case illustrated above, the existence of an adjoint to the quotient guaranteed that the localization had small hom-sets. If we want to apply Theorem 1.20 to the quotient functor, we are forced to suppose this fact. This is not particularly stringent, since several criteria for the smallness of the hom-sets of a Verdier quotient exist, see for example [LO10, Theorem 1.21] and the references there.

Proposition 1.33. *Let $\mathcal{T}$ be a compactly generated triangulated category admitting small coproducts. Let $\mathcal{S} \subseteq \mathcal{T}$ be a localizing subcategory, and assume that the quotient ${}^{\mathcal{T}}\!/_{\mathcal{S}}$ is a category (i.e. has small hom-sets). Denote with $\pi \colon \mathcal{T} \to {}^{\mathcal{T}}\!/_{\mathcal{S}}$ the quotient functor. Then the following hold:*

a) *The quotient functor π admits a fully faithful right adjoint μ;*

b) *If for every compact object* $Y \in \mathcal{T}$ *the object* $\pi(Y)$ *is compact in* $\mathcal{T}/\mathcal{S}$*, then* μ *preserves coproducts.*

c) *If for every compact object* $Y \in \mathcal{T}$ *the object* $\pi(Y)$ *is compact in* $\mathcal{T}/\mathcal{S}$ *and* $\mathcal{T}$ *is compactly generated by a set* $\mathcal{R} \subseteq \mathcal{T}^c$*, then* $\mathcal{T}/\mathcal{S}$ *is generated by* $\pi(\mathcal{R})$*.*

Proof. Since $\mathcal{S}$ is localizing, π preserves coproducts; therefore, by Theorem 1.20 it admits a right adjoint μ. By Proposition 1.32, μ is fully faithful. This proves a). To prove b), observe that if $\pi(Y)$ is compact we get

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(Y, \mu(\bigoplus_i X_i)) \cong \mathrm{Hom}_{\mathcal{T}}(\pi(Y), \bigoplus_i X_i) \cong \bigoplus_i \mathrm{Hom}_{\mathcal{T}}(\pi(Y), X_i) \cong$$
$$\cong \bigoplus_i \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(Y, \mu(X_i)) \cong \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(Y, \bigoplus_i \mu(X_i))$$

for all $\{X_i\} \subseteq \mathcal{T}/\mathcal{S}$. Now Lemma 1.18 shows that the natural morphism

$$\bigoplus_i \mu(X_i) \to \mu(\bigoplus_i X_i)$$

is an isomorphism. Finally, to prove c), observe that for any $X \in \mathcal{T}/\mathcal{S}$, if

$$\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(\pi(Y), X[n]) = 0$$

for any $Y \in \mathcal{R}$ and $n \in \mathbb{Z}$, then

$$\mathrm{Hom}_{\mathcal{T}}(Y, \mu(X)[n]) = 0$$

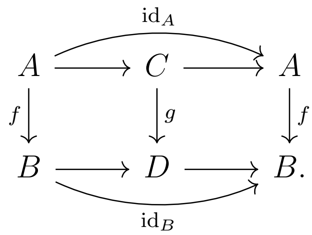
for any $Y \in \mathcal{R}$ and $n \in \mathbb{Z}$. Since $\mathcal{R}$ generates $\mathcal{T}$, this implies that $\mu(X) = 0$. Since μ is fully faithful, $X = 0$. □

1.4 Model categories

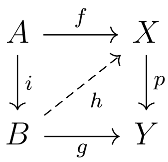
A different, and much more far reaching, approach to the theory of localizations is via Quillen's model categories. Roughly, a model structure on a category $\mathcal{C}$ is the datum of three subclasses of morphisms: weak equivalences, fibrations (that can be considered "well behaved surjections") and cofibrations ("well behaved injections") all of which have to satisfy some properties. Having a model structure on a category will allow us (among other things) to give a very explicit description of the localization of $\mathcal{C}$ at the weak equivalences, guaranteeing that the hom-spaces form a small set.

In this section, $\mathcal{C}$ will represent a fixed complete and cocomplete category.

Definition 1.34. A morphism f in $\mathcal{C}$ is a retract of a morphism g if there exists a commutative diagram of the form



Definition 1.35. Suppose that $i\colon A \to B$ and $p\colon X \to Y$ are morphisms in $\mathcal{C}$. We say that i has the left lifting property with respect to p and p has the right lifting property with respect to i if for any diagram of the type



there exists a morphism $h\colon B \to X$ making the whole diagram commute.

Definition 1.36. A model structure on $\mathcal{C}$ is defined as the datum of three classes of morphisms called fibrations, cofibrations and weak equivalences satisfying the following conditions:

1. If f and g are two composable morphisms such that two of f, g and gf are weak equivalences, so is the third;

2. Fibrations, cofibrations and weak equivalences are closed under retracts;

3. Call acyclic fibrations (some texts call these trivial fibrations) maps that are both fibrations and weak equivalences, and acyclic cofibrations maps that are both cofibrations and weak equivalences. Then, we require acyclic cofibrations to have the left lifting property with respect to fibrations and cofibrations to have the right lifting property with respect to acyclic fibrations;

4. Any morphism $A \xrightarrow{f} B$ can be factored in two ways; either as
$$A \xrightarrow{i} X \xrightarrow{p} B$$
where i is a cofibration and p is an acyclic fibration, or as
$$A \xrightarrow{i'} X \xrightarrow{p'} B$$
where i' is an acyclic cofibration and p' is a fibration.

It is customary (as seen for example in [Hov07]) to require for the factorization of 4. to be functorial in an appropriately defined sense. This does not follow from the other axioms, but is a very mild conditions and to this day are not known examples where it does not hold. We do not define it in detail, but we implicitly suppose it, as will be clear later. A model category is a (complete and cocomplete) category together with the choice of a model structure.

Example 1.4. The prototypical example of a model category is the category **Top** of topological spaces, with the weak equivalences taken to be the weak homotopy equivalences, fibrations the Serre fibrations and cofibrations the maps having the left lifting properties with respect to acyclic fibrations.

Remark. Those shown here were not Quillen's original axioms. Besides the functorial factorizations, Quillen only required finite limits and colimits to exist. Furthermore, the axioms we gave here were considered by Quillen the axioms of a *closed* model category. In the following years, the distinction between closed and general model categories has been dropped, with the recent authors giving the same definitions that we gave.

Since a model category admits all limits and colimits, in particular it admits an initial object 0 and a final object $*$.

Definition 1.37. An object X in a model category is said to be fibrant if the unique morphism $X \to *$ is a fibration. X is said to be cofibrant if the unique morphism $0 \to X$ is a cofibration.

Thanks to condition 4, by factoring the morphism $X \to *$ as

$$X \to X' \to *$$

where the first arrow is an acyclic cofibration and the second is a fibration we get, for any object X, an fibrant object X' with a weak equivalence $X' \to X$. By using the functorial factorizations, we get a fibrant replacement functor

$$\begin{aligned} \mathcal{C} &\to \mathcal{C} \\ X &\to RX \end{aligned}$$

that assigns to each object X an object RX that admits a weak equivalence $X \to RX$. Dually, by factoring the map $0 \to X$, we get a cofibrant replacement functor G, assigning to each object Y a cofibrant object QY that admits a weak equivalence $Y \to QY$. We will call QX and RX, together with the weak equivalences $X \to QX$ and $X \to RX$ cofibrant and fibrant replacements for X.

Denote with $\mathcal{W}$ the class of weak equivalences. A model structure on $\mathcal{C}$ allows us to give a very concrete description of the localized category $\mathcal{C}[\mathcal{W}^{-1}]$. This generalizes the case of the category **Top**: to localize it at the weak equivalences, one first uses the existence of CW approximations to substitute any space with a weakly equivalent CW complex. Then, one uses Whitehead's theorem to prove that, since any weak equivalence between CW complexes is a homotopy equivalence, the category localization of **Top** can be realized as the category of CW complexes where morphisms are homotopy classes of continuous maps. Here, we have used "can be realized" in a not very precise sense: the category obtained in this way is not exactly the localization of **Top** at the quasi-equivalences, since it has strictly less objects. It is, however, equivalent to it.

We want to give a similar process for an arbitrary model category $\mathcal{C}$. In this case, the role of "good objects" (i.e. CW complexes) will be taken by bifibrant objects, objects that are both fibrant and cofibrant. It is less clear what the correct notion of homotopy equivalence of morphisms is.

Definition 1.38. Let B be an object of $\mathcal{C}$. A cylinder object$B \wedge I$ for B is a factorization of the natural "folding" map

$$B \coprod B \to B$$

in a cofibration

$$B \coprod B \to B \wedge I$$

followed by a weak equivalence

$$B \wedge I \to B.$$

Dually, a path objectB^I for B if a factorization of the natural "diagonal" map

$$B \to B \times B$$

into a weak equivalence

$$B \to B^I$$

followed by a fibration

$$B^I \to B \times B.$$

By condition 4 in the definition of model category, it follows that cylinder and path objects always exist.

Definition 1.39. Let $f, g \colon X \to Y$ be morphisms in a model category. A left homotopy from f to g is a morphism $H \colon X \wedge I \to Y$ for some cylinder object

$X \wedge I$ for X such that, calling i_0 and i_1 the two natural maps $X \to X \coprod X$, the composition

$$X \xrightarrow{i_0} X \coprod X \to X \wedge I \xrightarrow{H} Y$$

equals f and the composition

$$X \xrightarrow{i_1} X \coprod X \to X \wedge I \xrightarrow{H} Y$$

equals g. Dually, a right homotopy between f and g is a map $K\colon X \to Y^I$ for some path object Y^I for Y such that, calling p_0 and p_1 the two natural projections $Y \times Y \to Y$, the composition

$$X \xrightarrow{K} Y^I \to Y \times Y \xrightarrow{p_0} Y$$

equals f and the composition

$$X \xrightarrow{K} Y^I \to Y \times Y \xrightarrow{p_1} Y$$

equals g.

Two morphisms f and g are said to be homotopic if they are both left and right homotopic; in that case, we write $f \sim g$. A morphism $X \xrightarrow{f} Y$ is a homotopy equivalence if there exists a morphism $Y \xrightarrow{g} X$ such that $fg \sim \mathrm{id}_Y$ and $gf \sim \mathrm{id}_X$. In the case of topological spaces, both left and right homotopy coincide with the usual definition of homotopy between continuous maps. In the general case, those are not always so well-behaved and in order to use them meaningfully we have to restrict the objects we apply them to.

Proposition 1.40. *Suppose that X, Y are bifibrant objects. Then the relations of left and right homotopy on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ coincide, and are an equivalence relation compatible with the composition, in the sense that if $f \sim g$ then $fh \sim gh$ and $lh \sim lg$ for any f, l composable with f and g.*

Proof. [Hov07, Proposition 1.2.5] □

Proposition 1.41. *A morphism $X \xrightarrow{f} Y$ between bifibrant objects is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. [Hov07, Proposition 1.2.8] □

Denote with $\mathcal{C}_{cf}$ the full subcategory of $\mathcal{C}$ spanned by bifibrant objects. By Proposition 1.40, we can define the category $\mathcal{C}_{cf}/\sim$ having the same objects of $\mathcal{C}_{cf}$ and homotopy classes of morphisms in $\mathcal{C}$ as morphisms. As a corollary of Proposition 1.41, we get

Corollary 1.42. *There is an equivalence of categories*

$$j\colon \mathcal{C}_{cf}/\sim \to \mathcal{C}_{cf}[\mathcal{W}^{-1}].$$

where j can be taken to be the identity on objects.

Proof. [Hov07, Corollary 1.2.9] □

The existence of the replacement functors gives the following

Proposition 1.43. *The inclusion*

$$\mathcal{C}_{cf} \hookrightarrow \mathcal{C}$$

induces an equivalence of categories

$$\mathcal{C}_{cf}[\mathcal{W}^{-1}] \xrightarrow{\sim} \mathcal{C}[\mathcal{W}^{-1}].$$

Proof. [Hov07, Theorem 1.2.10] □

Therefore, the localized category $\mathcal{C}[\mathcal{W}^{-1}]$ is equivalent to the category $\mathcal{C}_{cf}/\sim$ which, in particular, has small hom-sets (i.e. is a category according to our convention). As in the case of topological spaces, one has to be wary of the fact that $\mathcal{C}_{cf}/\sim$ has strictly less objects than $\mathcal{C}[\mathcal{W}^{-1}]$, and to obtain an explicit description of the hom-spaces of $\mathcal{C}[\mathcal{W}^{-1}]$ one has to choose a quasi-inverse to the equivalence induced by the inclusion. This can be done, for example, using the replacement functors.

Proposition 1.44. *There are natural isomorphisms*

$$\mathrm{Hom}_{\mathcal{C}}(QRX, QRY)/\sim \;\cong \mathrm{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)/\sim$$

for any $X, Y \in \mathcal{C}$.

Proof. This is again [Hov07, Theorem 1.2.10] □

This implies that any morphism in $\mathcal{C}[\mathcal{W}^{-1}]$ between two objects can be represented either via a zig-zag

$$X \to RX \leftarrow QRX \xrightarrow{f} QRY \to RY \leftarrow Y$$

for some morphism f, or via a zig-zag

$$X \leftarrow QX \to RQX \xrightarrow{g} RQY \leftarrow QY \to Y$$

for some morphism g. Note that, although we are representing morphisms via zig-zags, all the arrows except the middle one depend only on X and Y, so the class of all possible zig-zags between two fixed objects forms a small set.

A case that will be of interest to us is that where all the objects in $\mathcal{C}$ are fibrant; in that case, the functor R can be taken to be the identity and any morphism in $\mathcal{C}[\mathcal{W}^{-1}]$ can be represented via a zig-zag of the form

$$X \leftarrow QX \xrightarrow{f} QY \rightarrow Y$$

or, composing the two arrows, as a roof

$$X \leftarrow QX \xrightarrow{f'} Y$$

where $X \leftarrow QX$ is a cofibrant replacement for X, so in particular a quasi-equivalence.

1.5 Notes for Chapter 1

The calculus of fractions in relation to the localization of a category was introduced by Gabriel and Zisman in [GZ67]; Triangulated categories were first developed by Verdier in his thesis [Ver96] for their applications in algebraic geometry; at the same time, similar concepts appeared in algebraic topology from the work of Puppe. Their study was later systematized by Neeman in [Nee01b]. Model categories were invented by Quillen (see [Qui67]) as tools in homotopy theory: a very nice introduction is the paper [DS95], while a more comprehensive source is [Hov07].

Chapter 2

Interlude

There are several reasons for which oftentimes triangulated categories are not an optimal environment to work in. One of the main issues - and the cause of most of the others - is related to the fact that the cone of a morphism cannot be defined in a functorial way. In fact, it is not hard to see that as soon as a triangulated category is "interesting" (that is, non-abelian) it is never possible to define a cone functor. Postponing the formal definitions, the above comment is made precise by the following fact:

Proposition 2.1. *Let $\mathcal{T}$ be an idempotent complete triangulated category. If $\mathcal{T}$ admits functorial cones, then $\mathcal{T}$ is split abelian i.e., an abelian category in which all exact sequences split.*

There are several constructions that are not possible without functorial cones: for example, it is not possible to give a sensible triangulated structure on a functor category with triangulated target. In fact, letting $\mathcal{S}$ be any category and $\mathcal{T}$ a triangulated category, giving a triangulated structure to $\mathrm{Fun}(\mathcal{S}, \mathcal{T})$ would imply giving a notion of a cone of a natural transformation $F \xrightarrow{\eta} G$ between two functors. The only way to exploit the triangulated structure of $\mathcal{T}$ to define such an object is to consider, for any object $x \in \mathcal{S}$, the induced morphism

$$F(X) \xrightarrow{\eta_x} G(X)$$

and to complete it to a triangle

$$F(X) \xrightarrow{\eta_x} G(X) \to \mathrm{Cone}(\eta_x).$$

However, since the assignment of the cone was not functorial to begin with, this construction does not give in general a well defined functor $\mathrm{Cone}(\eta) \colon \mathcal{S} \to \mathcal{T}$: in particular, given a morphism $x \xrightarrow{f} y$ there is no natural morphism[1] $\mathrm{Cone}(\eta_x) \to \mathrm{Cone}(\eta_y)$. This difficulty in giving $\mathrm{Fun}(\mathcal{S}, \mathcal{T})$ a triangulated

[1] A morphism always exists, but it is not unique nor the choice can be made canonical in any way.

structure is not due to lack of imagination: even in very simple cases, such a structure may not exist. This is exemplified by the proposition:

Proposition 2.2. *Let $\mathcal{A}$ be a category and suppose that $\mathcal{A}$ has a zero object. If* $\mathrm{Mor}(\mathcal{A})$ *admits a triangulated structure, then $\mathcal{A} = 0$. In particular, the category of morphism in a triangulated category never admits a triangulated structure.*

The conditions imposed on $\mathcal{A}$ are minimal, as the proposition works even without supposing $\mathcal{A}$ to be additive: this is less surprising that it may seem, since in the hypotheses of the proposition $\mathcal{A}$ can be realized as a full subcategory of $\mathrm{Mor}(\mathcal{A})$ via the functor that assigns to an object a the morphisms $a \to 0$, thus always inheriting from $\mathrm{Mor}(\mathcal{A})$ an additive structure.

Similarly, it is in general very hard to define a reasonable notion of tensor product of triangulated categories, as noted in [BLL04].

The rest of this section is devoted to proving and expanding on those two propositions. This is not integral to the rest of the thesis, so the reader can feel free to skip it.

2.1 Failure of $\mathrm{Mor}(\mathcal{A})$ to have a triangulated structure

Let's begin our discussion with an easy lemma.

Lemma 2.3. *In a triangulated category, all epimorphisms split. That is, given an epimorphism*

$$f\colon A \to B$$

in a triangulated category $\mathcal{T}$, there exists a morphism $g\colon B \to A$ such that the composition

$$B \xrightarrow{g} A \xrightarrow{f} B$$

is equal to the identity.

Proof. Complete f to an exact triangle $A \xrightarrow{f} B \xrightarrow{\pi} C \to A[1]$. Since $\pi f = 0$ and f is an epimorphism, $\pi = 0$. Applying now the homological functor $\mathrm{Hom}(B, -)$ we get an exact sequence of abelian groups

$$\mathrm{Hom}(B, A) \xrightarrow{f_*} \mathrm{Hom}(B, B) \xrightarrow{\pi_*} \mathrm{Hom}(B, C).$$

Since $\pi = 0$ it follows that $\pi_* = 0$, and therefore f_* is surjective. At this point it is enough to observe that the map f_* is defined as the composition with f, and that then any morphism $g \in \mathrm{Hom}(B, A)$ whose image is the identity of B has the desired property. □

Definition 2.4. Let $\mathcal{A}$ be a category. The category $\mathrm{Mor}(\mathcal{A})$ is the category whose objects are triples (x, y, f) such that $x, y \in \mathrm{Ob}(\mathcal{A})$, $f \in \mathrm{Hom}(x, y)$ and whose morphisms between two objects (x, y, f) and (x', y', f') are commutative squares

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ x' & \xrightarrow{f'} & y' \end{array}$$

Equivalently, $\mathrm{Mor}(\mathcal{A})$ can be defined as the functor category $\mathrm{Fun}(\bullet \to \bullet, \mathcal{A})$, where $\bullet \to \bullet$ is the category with two objects and only one non-identity morphism between them.[2] The category $\bullet \to \bullet$ is known as walking arrow, or interval category.

We can now prove Proposition 2.2.

Proof or Proposition 2.2. Let a be an object of $\mathcal{A}$. Consider the diagram

$$\begin{array}{ccc} a & \xrightarrow{\mathrm{id}_a} & a \\ {\scriptstyle\mathrm{id}_a}\downarrow & & \downarrow \\ a & \longrightarrow & 0 \end{array}$$

In the category $\mathrm{Mor}(\mathcal{A})$, it represents a morphism α between the objects $a \stackrel{\mathrm{id}}{\to} a$ and $a \to 0$; α is an epimorphism, since both of its components are. Then if $\mathrm{Mor}(\mathcal{A})$ admits a triangulated structure, by Lemma 2.3 there exists a morphism β between $a \to 0$ and $a \stackrel{\mathrm{id}}{\to} a$ such that $\alpha\beta = Id$. Explicitly, this means that there exists a commutative diagram

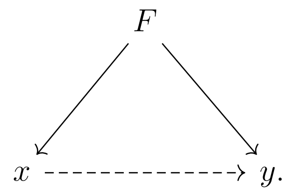
$$\begin{array}{ccc} a & \longrightarrow & 0 \\ {\scriptstyle f}\downarrow & & \downarrow \\ a & \xrightarrow{\mathrm{id}_a} & a \\ {\scriptstyle\mathrm{id}_a}\downarrow & & \downarrow \\ a & \longrightarrow & 0 \end{array}$$

such that the composition of both columns are the identities of a and 0 respectively. This implies that the identity morphism of a factors through the zero object and that thus $a = 0$. Since a was arbitrary, $\mathcal{A} = 0$. $\square$

[2] The definition is not symmetric: there exists one morphism from the first object to the second, but none from the second to the first.

2.2 Non-functoriality of cones

Definition 2.5. Let $\mathcal{C}$ be a category, $\mathcal{I}$ a small category and $F: \mathcal{I} \to \mathcal{C}$ a functor, seen as diagram of shape $\mathcal{I}$ in $\mathcal{C}$. A weak colimit for F is a cocone[3] $F \to x$ under the diagram F such that for any other cocone $F \to y$ there exists a (not necessarily unique) morphism $x \to y$ through which $F \to y$ factors.



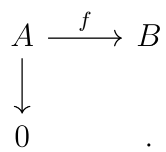
In other words, a weak colimit is an object that satisfies the existence property of a colimit but not necessarily the uniqueness. Dually, it is possible to define a weak limit.

[3]Here by cocone under a diagram we mean a natural transformation between the functor F and the constant functor Δ_x, and not anything related to cones in a triangulated category.

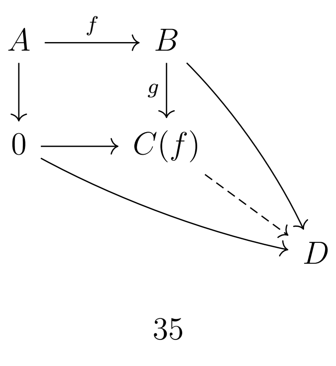
Although in practice we will ignore this fact, it is important to underline that in general a weak colimit is not unique, and two weak colimits of the same diagram may not be isomorphic.

Example 2.1. A weakly initial object is the weak colimit of the empty diagram. Concretely, a weakly initial object is an object x such that for any object y there exists a (not necessarily unique) morphism $x \to y$.

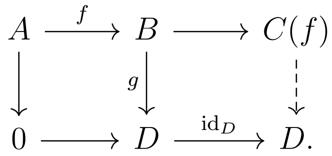
Example 2.2. Let $\mathcal{T}$ be a triangulated category. Recall that we had defined the cone $C(f)$ of a morphism $f: A \xrightarrow{f} B$ as any object $C(f)$ with morphisms $B \to C(f)$ and $C(f) \to A[1]$ such that $A \xrightarrow{f} B \to C(f) \to A[1]$ is an exact triangle. We affirm that $B \to C(f)$ is a weak cokernel of f, i.e. a weak colimit of the diagram



This means that for any commutative diagram



(that is, for any morphism $B \xrightarrow{g} C$ such that $gf = 0$) there exists a compatible morphism $C(f) \to D$. This follows easily from axiom TR3: consider the diagram

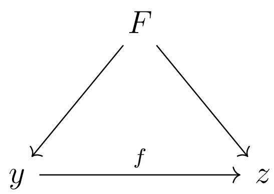


Since the composition $A \xrightarrow{f} B \xrightarrow{g} D$ vanishes the first square commutes, and since both rows can be completed to an exact triangle, by axiom TR3 there exists a compatible morphism $C(f) \to D$. This proves that $C(f)$ is a weak cokernel.

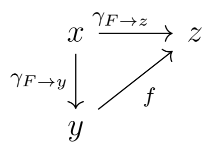
Notice that in most cases the induced morphism is in fact not unique, although as we have seen, two cones of the same morphism are always (not canonically) isomorphic.

Dually, a shifted cone $C(f)[-1] \to A$ is easily seen to be a weak kernel.

Definition 2.6. A functorial weak limit (x, γ) of a diagram F is a weak colimit $F \to x$ equipped with the choice for any cocone $F \to y$ of a morphism $\gamma_{F \to y} \colon x \to y$ factoring $F \to y$ such that the choice of factorization is functorial: for any morphism of cocones



the chosen factorizations make the triangle below commute.



Again, one can give the dual definition of a functorial weak limit.

Definition 2.7. A category $\mathcal{C}$ is said to be idempotent complete if all idempotents in $\mathcal{C}$ split: that is, for any morphism $e \colon x \to x$ such that $e^2 = e$ there exist an object y and a factorization $x \xrightarrow{r} y \xrightarrow{s} x$ such that $sr = e$ and $rs = \mathrm{id}_y$.

Example 2.3. The category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$ is idempotent complete: for any endomorphism of a vector space $T\colon V \to V$ such that $T^2 = T$ the factorization $V \xrightarrow{T} \mathrm{Im}(T) \hookrightarrow V$ gives the desired splitting. Similarly, all abelian categories are idempotent complete.

The notion of functorial weak colimit is an intermediate one between that of weak colimit and of colimit: the uniqueness of the factorization always implies its functoriality. However, as the next propositions shows, a functorial weak colimit is much closer to a colimit in the classical sense than a weak colimit.

Proposition 2.8. *Let $\mathcal{C}$ be an idempotent complete category. If $\mathcal{C}$ admits a functorial weakly initial object then $\mathcal{C}$ admits an initial object.*

Proof. Let (i, γ) be a functorial weakly initial object. By definition i is equipped with a morphism $i \xrightarrow{\gamma_i} i$. To begin with, we shall prove that if $\gamma_i = \mathrm{id}_i$ then i in initial. Indeed, for any $c \in \mathcal{C}$ and for any morphism $i \xrightarrow{\alpha} c$ the triangle

$$\begin{array}{ccc} i & \xrightarrow{\gamma_i} & i \\ {\scriptstyle\gamma_c}\downarrow & \swarrow{\scriptstyle\alpha} & \\ c & & \end{array}$$

is commutative. Then, if γ_i is the identity, $\alpha = \gamma_c$ and i is initial. In order to deal with the general case, we see that by the commutativity of the triangle

$$\begin{array}{ccc} i & \xrightarrow{\gamma_i} & i \\ {\scriptstyle\gamma_i}\downarrow & \swarrow{\scriptstyle\gamma_i} & \\ i & & \end{array}$$

γ_i is idempotent. Therefore there exist an object i' and a splitting $i \xrightarrow{a} i' \xrightarrow{b} i$ such that $ab = \mathrm{id}_{i'}$ and $ba = \gamma_i$. We will now prove that i' is an initial object. It is clearly a weakly initial object, and provided with the choice of factorizations $\delta_i = \gamma_i b$ it becomes a functorial weakly initial object as well. By considering the diagram

$$\begin{array}{ccc} i' & \xrightarrow{\delta_i} & i \\ {\scriptstyle\delta_{i'}}\downarrow & \swarrow{\scriptstyle a} & \\ i' & & \end{array}$$

we get that $\delta_{i'} = a\delta_i$. Since δ_i is by definition equal to $\gamma_i b$ and that $\gamma_i = ba$, we find $\delta_{i'} = a\delta_i = a\gamma_i b = abab = \mathrm{id}_{i'}$, therefore i' is an initial object. $\square$

Corollary 2.9. *Suppose $\mathcal{C}$ is an idempotent complete category, $\mathcal{I}$ a small category and $F\colon \mathcal{I} \to \mathcal{C}$ a diagram of shape $\mathcal{I}$. If $\mathcal{C}$ admits a functorial weak colimit (x, γ) of F than C admits a colimit of F.*

Proof. The category of cocones under F inherits from $\mathcal{C}$ the property of being idempotent complete: at this point, the claim follows from Lemma 2.8 and the fact that a functorial weak colimit is a functorial weakly initial object in the category of cocones under F while a colimit is an initial object in the same category. □

Of course, the same is true for limits. Let us now give the precise definition of what means for a triangulated category to have functorial cones (in order to show that they usually don't exist).

Definition 2.10. Let $\mathcal{T}$ be a triangulated category. The category of exact triangles in $\mathcal{T}$ is the category $\tau(\mathcal{T})$ whose objects are exact triangles and whose morphisms are morphisms of triangles.

Definition 2.11. A cone functor in a triangulated category is a functor

$$C\colon \operatorname{Mor}(\mathcal{T}) \to \tau(\mathcal{T})$$

of the form

$$C(X \xrightarrow{f} Y) = X \xrightarrow{f} Y \xrightarrow{a(f)} K(f) \xrightarrow{b(f)} X[1]$$

such that for any morphism in $\operatorname{Mor}(\mathcal{T})$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

the induced diagram is of the form

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{a(f)} & K(f) & \xrightarrow{b(f)} & X[1] \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle C_K^{h,k}} & & \downarrow{\scriptstyle h[1]} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{a(f')} & K(f') & \xrightarrow{b(f')} & X'[1] \end{array}$$

for some $C_K^{h,k}\colon K(f) \to K(f')$. A triangulated category is said to admit functorial cones if a cone functor exists.

Of course the object $K(f)$ and the morphisms $a(f)$ and $b(f)$ together with the morphism $C_K^{h,k}$ define functors to appropriate categories as well, and the functor C can be reconstructed from those. While we defined a cone functor as a whole triangle, if it will be clear from the context we will refer to either the object $K(f)$, the morphism $Y \overset{a(f)}{\to} K(f)$, or even the full triangle as the cone of f.

We can now give the proof of Proposition 2.1.

Proof of Proposition 2.1. The proof is slightly long but very straightforward, and is a basic consequence of Proposition 2.2. Let's first prove that if $\mathcal{T}$ admits functorial cones then it admits kernels and cokernels. Suppose that there exists a cone functor $C\colon \operatorname{Mor}(\mathcal{T}) \to \tau(\mathcal{T})$. Suppose for simplicity also that

$$C(0 \to A) = 0 \to A \overset{\text{id}}{\to} A \to 0$$

for every object A; this is not restrictive since any two cones are always isomorphic. Let us now prove that $Y \overset{a(f)}{\to} K(f)$ is in fact a functorial weak cokernel, which will imply that it is a cokernel. We have already seen in Example 2.2 that it is a weak cokernel, so we only need to see that it is functorial according to Definition 2.6. Following Example 2.2, we see that for any morphism $B \overset{g}{\to} D$ such that $gf = 0$ we get a morphism in $\operatorname{Mor}(\mathcal{T})$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & D \end{array}$$

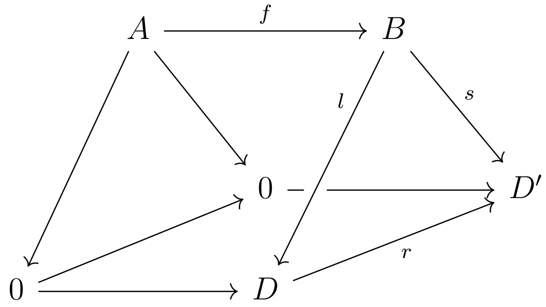
which, through the functor C, induces a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{a(f)} & K(f) \\ \downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle C_K^{0,g}} \\ 0 & \longrightarrow & D & \xrightarrow{\text{id}_D} & D. \end{array}$$

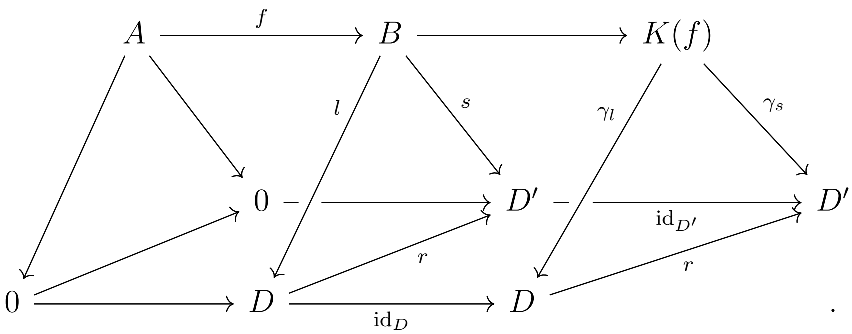
Define factorizations (as in Definition 2.6) $\gamma_g\colon K(f) \to D$ equal to $C_K^{0,g}$. The claim now reduces to proving that for two given morphisms $l\colon B \to D$ and $s\colon B \to D'$ such that $lf = sf = 0$ and any morphism $r\colon D \to D'$ such that $s = rl$ the induced triangle

$$\begin{array}{ccc} & K(f) & \\ {\scriptstyle \gamma_l}\swarrow & & \searrow{\scriptstyle \gamma_s} \\ D & \xrightarrow[r]{} & D' \end{array}$$

commutes. Consider the commutative diagram



that represents a commutative triangle in $\mathrm{Mor}(\mathcal{T})$, by considering the horizontal arrows as vertices of the triangle. Applying the functor C to such a triangle we get the diagram



The commutativity of the rightmost triangle gives what we were looking for. We have then proved that $\mathcal{T}$ has cokernels and, dually, that it has kernels. At this point, since by Proposition 2.3 all epimorphisms and monomorphisms in $\mathcal{T}$ split it follows that all monomorphisms are kernels and all epimorphisms are cokernels, so $\mathcal{T}$ is abelian. By the splitting lemma, since all monomorphisms (and all epimorphisms, but only one of those classes is sufficient) split, all exact sequences split. □

Here the condition for $\mathcal{T}$ to be idempotent complete is necessary, but not very restrictive; it can be proved that any triangulated category that admits countable coproducts is idempotent complete ([Nee01b, Proposition 1.6.8]), and that any triangulated category can be realized as a full triangulated subcategory of an idempotent complete triangulated category.

2.3 Notes for Chapter 2

Proposition 2.1 has been known since the introduction of triangulated categories, in particular being proven in Verdier's original thesis ([Ver96, Proposition 1.2.13]). This particular proof, only requiring the category to have split idempotents instead of countable coproducts is taken from the short note [Ste18]. Proposition 2.2 comes from this mathoverflow answer, and apparently is due to Paul Balmer.

Chapter 3

Dg-categories

3.1 Elements of enriched category theory

In this section we mainly give some useful definition. We begin with the definition of a monoidal category. Roughly speaking, a monoidal category is a category endowed with a sensible notion of a (functorial) tensor product.

Definition 3.1. A monoidal category is a category $\mathcal{K}$ equipped with the following extra data:

- a functor $- \otimes - \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, usually called tensor product;
- An object $1 \in \mathcal{K}$, called the identity object;
- a natural isomorphism $a\colon ((- \otimes -) \otimes -) \xrightarrow{\sim} (- \otimes (- \otimes -))$, called the associator;
- Two natural isomorphisms

$$\lambda\colon 1 \otimes - \xrightarrow{\sim} \mathrm{Id}$$

$$\rho\colon\ - \otimes 1\ \xrightarrow{\sim} \mathrm{Id};$$

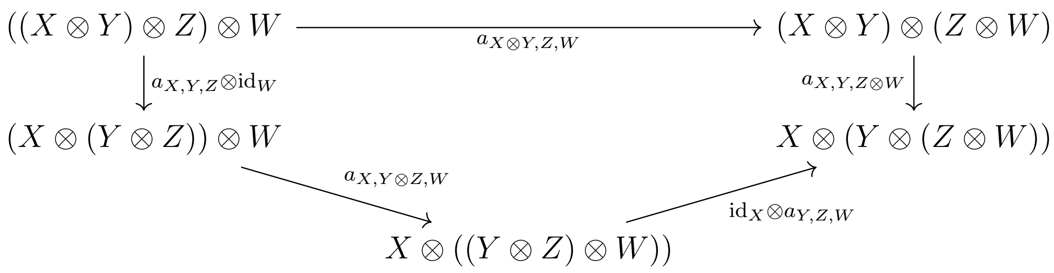
all satisfying the following compatibility conditions: denoting with

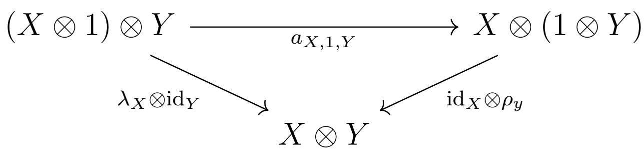
$$a_{X,Y,Z}\colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

$$\lambda_X\colon X \otimes 1 \xrightarrow{\sim} X$$

$$\rho_X\colon 1 \otimes X \xrightarrow{\sim} X$$

the components of the given natural transformations, we require the following diagrams to commute.





Example 3.1. The category $\mathbf{Vect}_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$, equipped with the tensor product of vector spaces and with $\mathbb{K}$ (considered as a vector space over itself) as the identity is a monoidal category.

Example 3.2. The category **Set** of sets, equipped with the cartesian product as a tensor product and with any singleton as the identity is a monoidal category.

Monoidal categories are useful because (among other things) they make it possible to define enriched categories, i.e. categories where the hom-spaces, instead of belonging to the category **Set**, belong to an arbitrary monoidal category; the existence of the tensor product allows to define an appropriate notion of compatibility between the composition of morphisms and the internal structure of the monoidal category: for example, when requiring the hom-spaces of a category to be vector spaces, one would certainly require the composition of morphisms to be linear in either variable.

Definition 3.2. Let $\mathcal{K}$ be a monoidal category. A category $\mathcal{C}$ enriched over $\mathcal{K}$ (or $\mathcal{K}$-category) is the datum of

- A collection of objects $\mathrm{Ob}(\mathcal{C})$;
- For every two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a *hom-object* $\mathcal{C}(X, Y) \in \mathrm{Ob}(\mathcal{K})$;
- For every three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ a morphism in $\mathcal{K}$

  $$\circ_{X,Y,Z} \colon \mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z),$$

  representing the composition of morphisms;

- For each object X in $\mathcal{C}$, a morphism $j_X : 1 \to \mathcal{C}(X,X)$, representing the identity morphism.

Satisfying the associativity and unity axioms expressed by the commutativity of the following diagrams:

$$\begin{array}{ccc}
(\mathcal{C}(Z,W)\otimes\mathcal{C}(Y,Z))\otimes\mathcal{C}(X,Y) & \xrightarrow{a_{\mathcal{C}(Z,W),\mathcal{C}(Y,Z),\mathcal{C}(X,Y)}} & \mathcal{C}(Z,W)\otimes(\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y)) \\
\downarrow{\scriptstyle \circ_{Y,Z,W}\otimes\mathrm{id}_{\mathcal{C}(X,Y)}} & & \downarrow{\scriptstyle \mathrm{id}_{\mathcal{C}(Z,W)}\otimes\circ_{X,Y,Z}} \\
\mathcal{C}(Y,W)\otimes\mathcal{C}(X,Y) & & \mathcal{C}(Z,W)\otimes\mathcal{C}(X,Z) \\
\searrow{\scriptstyle \circ_{X,Y,W}} & & \swarrow{\scriptstyle \circ_{X,Z,W}} \\
& \mathcal{C}(Z,W) &
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{C}(Y,Y)\otimes\mathcal{C}(X,Y) & \xrightarrow{\circ_{X,Y,Y}} & \mathcal{C}(X,Y) & \xleftarrow{\circ_{X,X,Y}} & \mathcal{C}(X,Y)\otimes\mathcal{C}(X,X) \\
\uparrow{\scriptstyle j_Y\otimes\mathrm{id}_{\mathcal{C}(X,Y)}} & \nearrow{\scriptstyle \rho_{\mathcal{C}(X,Y)}} & & \nwarrow{\scriptstyle \lambda_{\mathcal{C}(X,Y)}} & \uparrow{\scriptstyle \mathrm{id}_{\mathcal{C}(X,Y)}\otimes j_X} \\
1\otimes\mathcal{C}(X,Y) & & & & \mathcal{C}(X,Y)\otimes 1.
\end{array}$$

A $\mathcal{K}$-category $\mathcal{C}$ is said to be small if $\mathrm{Ob}(\mathcal{C})$ form a set (and not a proper class). The notion of $\mathcal{K}$-enriched category is very flexible; taking $\mathcal{K}$ to be respectively the category **Set** of sets, **Cat** of categories[1] , **Ab** of abelian groups and k-**Mod** of modules over a commutative ring k, we find the notions of ordinary category, (strict) 2-Category, preadditive category and k-linear category.

[1] Disregarding set-theoretic issues.

Not very relevant but interesting example. Recall that a partially ordered set can be considered as a category by letting the hom spaces be composed of a single arrow $x \to y$ if $x \leq y$ and none otherwise. Consider the partially ordered set $[0,\infty]$. Addition, extended in the obvious way to ∞, gives to $[0,\infty]$ the structure of a monoidal category, with 0 as the unit. A (small) category enriched over this monoidal category is then composed of:

- A set X;
- For any $x, y \in X$, a number $d(x,y)$, eventually infinite;
- For any $x, y, z \in X$, an arrow $d(x,y) + d(y,z) \to d(x,z)$; this means just that $d(x,y) + d(y,z) \leq d(x,z)$.

satisfying the unital and associative condition: associativity is trivially satisfied, while the unity condition reads: $d(x,x) = 0$ implies $x = 0$. The reader will then easily recognize that a category enriched over $[0,\infty]$ is nothing but a - slightly generalized to be non necessarily symmetric and to admit infinite distances - metric space.

It is now easy to give the definition of a $\mathcal{K}$-functor between $\mathcal{K}$-categories.

Definition 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{K}$-categories, for a given monoidal category $\mathcal{K}$. A $\mathcal{K}$-functor $\mathcal{T}\colon \mathcal{A} \to \mathcal{B}$ is the datum of a map $\mathcal{T}\colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$ together with, for any objects X, Y in $\mathrm{Ob}(\mathcal{A})$, a map $\mathcal{T}_{X,Y}\colon \mathcal{A}(X,Y) \to \mathcal{B}(X,Y)$ in $\mathcal{K}$, subject to the compositions and unity conditions expressed by the commutativity of the diagrams

$$\begin{array}{ccc}
\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) & \xrightarrow{\circ_{X,Y,Z}} & \mathcal{A}(X,Z) \\
{\scriptstyle \mathcal{T}_{Y,Z}\otimes \mathcal{T}_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mathcal{T}_{X,Z}} \\
\mathcal{B}(TY,TZ) \otimes \mathcal{B}(TX,TY) & \xrightarrow{\circ_{TX,TY,TZ}} & \mathcal{B}(TX,TZ)
\end{array}$$

$$\begin{array}{ccc}
 & & \mathcal{A}(X,X) \\
 & \nearrow^{j_X} & \\
1 & & \downarrow{\scriptstyle \mathcal{T}_{X,X}} \\
 & \searrow_{j_Y} & \\
 & & \mathcal{B}(TX,TX).
\end{array}$$

The compositions and identity morphisms of $\mathcal{A}$ and $\mathcal{B}$ have been denoted with the same symbol, but this should not create any confusion since there is no possible ambiguity.

One can also define a notion of $\mathcal{K}$-natural transformation between $\mathcal{K}$-functors (see for example [Kel82, Section 1.2]) but those are slightly more involved to define abstractly, so we will only define them in the case that is relevant to this thesis. Indeed, from this point on we will focus on a specific monoidal category, the category $\mathbf{C}(k)$ of chain complexes of k-modules.

3.1.1 The monoidal category $\mathbf{C}(k)$

From now on, we will let k be a fixed commutative ring with unity. We give some definitions in order to fix the notations.

Definition 3.4. A graded k-module is a k-module V together with a decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V^i.$$

We will call the elements $x \in V^n$ homogeneous elements of degree n.

Definition 3.5. A graded morphism of degree n between graded modules V and W is a morphism of k-modules $f\colon V \to W$ such that $f(V^i) \subseteq W^{i+n}$ for every $i \in \mathbb{Z}$.

Notice that, since a graded module is defined as a coproduct, the datum of a graded morphism between two graded modules coincides that that of the (infinite) collection of maps between homogeneous components.

Definition 3.6. A chain complex of k-modules is a graded k-module A together with a fixed morphism $d_A\colon A \to A$ of degree 1 such that $d^2 = 0$, called the differential. Of course, a chain complex of k-modules is nothing but a chain complex in the abelian category of k-modules.

When there is no risk of ambiguity we will often omit the subscript A when writing the differential, in order to simplify the notation. In the following when we write chain complex, unless something else is specified, we will always mean chain complex of k-modules. If A is a chain complex, we can define the associated graded modules

- $Z^*A = \operatorname{Ker} d \subseteq A$, whose homogeneous elements of degree n are called n-cycles.
- $B^*A = \operatorname{Im} d \subseteq A$, whose homogeneous elements of degree n are called n-boundaries.
- $H^*A = Z^*A/B^*A$, called the homology of A.

Of course, the grading of Z^*A and B^*A is induced by that of A and the definition of H^*A is made possible by the fact that $d^2 = 0$ (and hence $B^*A \subseteq Z^*A$).

Definition 3.7. A chain map between two chain complexes A, C is a degree 0 morphism of graded modules $f\colon A \to C$ such that $df = fd$. An isomorphism of chain complexes is a chain map that admits an inverse.

Remark. A chain map $f\colon A \to C$ restricts to a degree 0 morphism of graded modules

$$f\colon Z^*A \to Z^*C$$

such that $f(B^*A) \subseteq B^*C$. Therefore, it also induces a degree 0 morphism $H^*A \to H^*C$.

We will denote with $\mathbf{C}(k)$ the category whose objects are chain complexes and whose morphisms are chain maps, and with $\mathbf{Gr}(k)$ the category whose objects are graded modules and morphisms are degree 0 morphisms of graded modules.

Definition 3.8. A chain complex A is said to be acyclic if $H^*A = 0$. Similarly, a chain map $A \to B$ is a quasi-isomorphism if it induces an isomorphism $H^*A \to H^*B$.

Definition 3.9. The shifted chain complex $A[n]$ is defined as the complex having

$$A[n]^i = A^{n+i}$$

and

$$d_{A[n]} = (-1)^n d_A.$$

The construction is functorial, since a chain map $f\colon A \to B$ induces a natural chain map $f[n]\colon A[n] \to B[n]$ (defined tautologically). It is also clear by the definition that $Z^iA[j] = Z^{i+j}A$, $B^iA[j] = B^{i+j}A$ and $H^iA[j] = H^{i+j}A$. The functor $[n]$ is an isomorphism, its inverse being the functor $[-n]$.

Definition 3.10. Let A, B be chain complexes. Their tensor product $A \otimes B$ is the chain complex defined by

$$(A \otimes B)^n = \bigoplus_{i+j=n} A^i \otimes_k B^j$$

whose differential is defined on homogeneous elements as

$$d_{A\otimes B}(x \otimes y) = d_A x \otimes y + (-1)^{\deg(x)} x \otimes d_B y.$$

The tensor product of complexes is associative: given three complexes A, B and C, unwinding the definition gives

$$((A \otimes B) \otimes C)^n \cong \bigoplus_{i+j+k=n} A^i \otimes B^j \otimes C^k \cong (A \otimes (B \otimes C))^n$$

and, for homogeneous $x \in A^i, y \in B^j, z \in C$,

$$d_{(A\otimes B)\otimes C}(x \otimes y \otimes z) = dx \otimes y \otimes z + (-1)^i x \otimes dy \otimes z + (-1)^{i+j} x \otimes y \otimes dz =$$
$$= d_{A\otimes(B\otimes C)}(x \otimes y \otimes z)$$

for homogeneous x, y and z. Similarly, we define

$$(f \otimes g)(v \otimes w) = (-1)^{pq} f(v) \otimes g(v)$$

when g is a degree p morphism and v has degree q.

Similarly, by defining (by abuse of notation) the complex k as the complex having k in degree 0 and 0 elsewhere, it is immediate to verify that one can take the just defined tensor product of chain complexes, k as the identity object and endow $\mathbf{C}(k)$ with the structure of a monoidal category. Recall that a chain complex A is said to be concentrated in degree 0 if $A^i = 0$ for every $i \neq 0$, so k is a chain complex concentrated in degree 0.

$\mathbf{C}(k)$ has a way richer structure than just that of a monoidal category. An important feature is that it is a *closed* monoidal category: since the functor given by tensoring with an arbitrary object has a right adjoint, it is possible to define an internal hom, an object of the category itself representing the space of morphisms between two other objects. We will see how this will turn $\mathbf{C}(k)$ into a category enriched over itself. Let's see precisely what we mean by this.

Definition 3.11. Let A, B be chain complexes. Their internal hom $\mathcal{H}om(A, B)$ is the chain complex defined by

$$\mathcal{H}om(A, B)^n = \prod_i \mathrm{Hom}(A^i, B^{n+i}) = \{f^i \colon A^i \to B^{i+n}\}_{i \in \mathbb{Z}}$$

that is, elements of degree n are the morphisms of degree n between the A and B seen as graded k-modules (without any compatibility condition with the differential). The differential is defined on homogeneous elements $f \in \mathcal{H}om(A, B)^n$ as

$$df = d_B f - (-1)^n f d_A.$$

Remark. According to the definitions, the elements $f \in \mathcal{H}om(A, B)^0$ the maps respecting the grading, while those of $Z^0\mathcal{H}om(A, B)$ are the chain maps. The elements of $H^0\mathcal{H}om(A, B)$ are the chain maps *up to homotopy*: indeed, given two chain maps $f, g \in Z^0\mathcal{H}om(A, B)$, an homotopy between f and g is precisely a morphism $h \in \mathcal{H}om(A, B)^{-1}$ such that $dh = f - g$.

Proposition 3.12. *There is an isomorphism*

$$\mathcal{H}om(A[n], B) \cong \mathcal{H}om(A, B)[-n].$$

Proof. Take any $f \in \mathcal{H}om(A[n], B)^m$. Since $A[n]^i = A^{n+i}$, f is represented by a collection

$$f^i : A^{i+n} \to B^{i+m}.$$

To this we associate the morphism represented by $(-1)^n f^i$ considered as an element of $\mathcal{H}om(A, B)^{m-n} = \mathcal{H}om(A, B)[-n]^m$, that we denote $\tilde{f}$. It is immediate to verify that this is an isomorphism of graded modules. Let's check the differentials;

$$d_{\mathcal{H}om(A[n],B)}f = d_B f - (-1)^m f d_{A[n]} = d_B f - (-1)^m f d_A$$

while

$$d_{\mathcal{H}om(A,B)[-n]}\tilde{f} = (-1)^n((-1)^n d_B f - (-1)^{m-n} f d_A) = d_B f - (-1)^m f d_A.$$

□

In the same way, there is an isomorphism

$$\mathcal{H}om(A, B[n]) \cong \mathcal{H}om(A, B)[n].$$

Remark. The internal Hom defines a functor

$$\mathcal{H}om(-,-)\colon \mathbf{C}(k)^{op} \times \mathbf{C}(k) \to \mathbf{C}(k),$$

the action on morphisms being the usual (pre)composition.

Proposition 3.13. *For any complex B, The internal Hom functor $X \to \mathcal{H}om(B, X)$ is a right adjoint to the tensor functor $X \to X \otimes B$. This means that, for any $A, C \in \mathbf{C}(k)$, we have a natural isomorphism*

$$\mathrm{Hom}(A \otimes B, C) \cong \mathrm{Hom}(A, \mathcal{H}om(B, C)).$$

In general, we say that a monoidal category is closed if the functor defined by tensoring by a fixed object admits a right adjoint, and we will freely call internal hom the right adjoint. According to this definition, the proposition above states that $\mathbf{C}(k)$ is a closed monoidal category.

Proof. We actually prove a stronger result, that being the natural isomorphism (of chain complexes!)

$$\mathcal{H}om(A \otimes B, C) \cong \mathcal{H}om(A, \mathcal{H}om(B, C)).$$

The claim will then follow by considering the 0-cycles. This follows from unwinding the definitions: as a k-module,

$$\begin{aligned}\mathcal{H}om(A \otimes B, C)^n &= \prod_i \mathrm{Hom}((A \otimes B)^i, C^{i+n}) = \prod_i \mathrm{Hom}\left(\bigoplus_j A^j \otimes B^{i-j}, C^{i+n}\right) \cong \\ &\cong \prod_{i,j} \mathrm{Hom}(A^j \otimes B^{i-j}, C^{i+n}) \cong \prod_{i,j} \mathrm{Hom}(A^j, \mathrm{Hom}(B^{i-j}, C^{i+n})) \cong \\ &\cong \prod_i \mathrm{Hom}(A^j, \prod_j \mathrm{Hom}(B^{i-j}, C^{i+n})) \cong \\ &\cong \prod_i \mathrm{Hom}(A^j, \prod_l \mathrm{Hom}(B^l, C^{l+i+n})) \cong \mathcal{H}om(A, \mathcal{H}om(B, C))^n,\end{aligned}$$

where we have used the usual tensor-hom adjunction for k-modules, and in the end shifted the index of the product. One also shows that the differentials of these two complexes coincide: let's take, as usual, homogeneous elements $f \in \mathcal{H}om(A \otimes B, C)^n, v \in A^i$ and $w \in B$. Then,

$$\begin{aligned}(d_{\mathcal{H}om(A\otimes B,C)}f)(v \otimes w) = d_C(f(v \otimes w)) - (-1)^n f(d_{A\otimes B} v \otimes w) = \\ = d_C(f(v \otimes w)) - (-1)^n f(d_A v \otimes w) - (-1)^{n+i} f(v \otimes d_B w).\end{aligned}$$

On the other hand, let $f \in \mathcal{H}om(A, \mathcal{H}om(B, C))^n$ and v, w as above. Since $\deg(f(v)) = n + i$, we have

$$\begin{aligned}(d_{\mathcal{H}om(A,\mathcal{H}om(B,C))}f)(v)(w) = (d_{\mathcal{H}om(B,C)})(f(v))(w) - (-1)^n f(d_A v)(w) = \\ = d_C(f(v)(w)) - (-1)^{n+i} f(v)(d_B w) - (-1)^n f(d_A v)(w).\end{aligned}$$

Recalling that the isomorphisms (of k-modules)

$$\operatorname{Hom}(X \otimes Y, Z) \cong \operatorname{Hom}(X, \operatorname{Hom}(Y, Z))$$

is defined by sending $f : X \otimes Y \to Z$ to the morphism

$$x \to [y \to f(x \otimes y)],$$

it is clear that the two differentials coincide. □

By Yoneda's Lemma, the above proposition characterizes the tensor product of complexes in term of internal homs, and vice versa.

The stronger result we proved is in fact completely general.

Proposition 3.14. *Let $\mathcal{K}$ be a monoidal category. Suppose that for any $\mathcal{B} \in \mathcal{K}$ the functor*

$$A \to A \otimes B$$

admits a right adjoint, denoted as $[B, -]$. Then, for any three objects $A, B, C \in \mathcal{K}$ the adjunction isomorphism induces a natural isomorphism

$$[A \otimes B, C] \cong [A, [B, C]].$$

Proof. By definition of adjunction, we have for every $X \in \mathcal{K}$ isomorphisms

$$\begin{aligned}\operatorname{Hom}(X, [A \otimes B, C]) \cong &\operatorname{Hom}(X \otimes (A \otimes B), C]) \cong \operatorname{Hom}((X \otimes A) \otimes B, C) \cong \\ \cong &\operatorname{Hom}(X \otimes A, [B, C]) \cong \operatorname{Hom}(X, [A, [B, C]]).\end{aligned}$$

The claim then follows by Yoneda's Lemma. □

We have also proved that the category $\mathbf{Gr}(k)$ is monoidal closed, since all the definitions and proofs keep working in that case (simply ignoring everything that was said about the differentials).

3.2 Dg-categories

Definition 3.15. A (k-linear) dg-category is a category $\mathcal{A}$ enriched over the monoidal category $\mathbf{C}(k)$.

Explicitly, this means that a dg-category $\mathcal{A}$ is composed by:

- A class of objects $\mathrm{Ob}(\mathcal{A})$;
- For every objects $A, B \in \mathrm{Ob}(\mathcal{A})$, a chain complex of k-modules $\mathcal{A}(A, B)$;
- For each object $A \in \mathcal{A}$, a morphism $k \to \mathcal{A}(A, A)$. Recall that k is considered as a chain complex concentrated in degree 0;
- A composition law $\mathcal{A}(B, C) \otimes \mathcal{A}(A, B) \to \mathcal{A}(A, C)$ satisfying the obvious associativity and unit conditions. Notice that the composition is a morphism in $\mathbf{C}(k)$, i.e. a chain map.

Similarly, a graded category is a category enriched over the category $\mathbf{Gr}(k)$ of graded modules. A dg-category is also a graded category simply by forgetting the extra structure given by the differential.

A dg (or graded) category is said to be small if the class $\mathrm{Ob}(\mathcal{A})$ forms a set. We will call k the base ring of the category $\mathcal{A}$.

We say that a morphism $f \in \mathcal{A}(A, B)$ is closed if $df = 0$. Denoting with id_A the image of $1 \in k$ in $\mathcal{A}(A, A)$, it immediately follows from the definitions that id_A is closed of degree 0, since $k \to \mathcal{A}(A, A)$ is a chain map. Similarly, taking $g \in \mathcal{A}(A, B)^n$ and $f \in \mathcal{A}(B, C)^m$ it follows from the fact that the composition is a chain map (and thus respects the grading) and from the definition of tensor product of complexes that $f \circ g \in \mathcal{A}(A, C)^{n+m}$ and that the graded Leibniz rule holds:

$$d(f \circ g) = df \circ g + (-1)^{\deg(f)} f \circ dg.$$

A full dg-subcategory $\mathcal{B}$ of a dg-category $\mathcal{A}$ is a dg-category formed by a subclass of the objects of $\mathcal{A}$, and with $\mathcal{B}(A, B) = \mathcal{A}(A, B)$ for $A, B \in \mathrm{Ob}(\mathcal{B}) \subseteq \mathrm{Ob}(\mathcal{A})$. If $f \in Z^0\mathcal{A}(X, Y)$ is a closed degree 0 morphism, a nullhomotopy of f is a morphism $h \in \mathcal{A}(X, Y)^{-1}$ such that $dh = f$.

Example 3.3. The category $\mathbf{C}(k)$ admits the structure of a dg-category, with the internal homs taken as hom object. This amounts to saying that we can define a composition map

$$\mathcal{H}om(B, C) \otimes \mathcal{H}om(A, B) \to \mathcal{H}om(A, C)$$

that is a chain map and that respects the conditions of definition 3.2: this is given by the usual composition of morphisms.

We will denote with $\mathbf{C}_{\mathrm{dg}}(k)$ the category $\mathbf{C}(k)$ seen as a dg-category. Similarly, given a k-linear additive category A, we can define the category $\mathbf{C}_{\mathrm{dg}}(A)$ of complexes of objects of A, with internal hom defined as in the case of k-modules.

Remark. The above example is manifestation of a general fact: any closed monoidal category $\mathcal{K}$ admits an enrichment over itself. Denote as usual with $[X, -]$ the right adjoint to $- \otimes X$. First of all observe that the adjunction defines, for every objects $X, Y \in \mathcal{K}$, a canonical morphism

$$[X, Y] \otimes X \xrightarrow{\mathrm{ev}} Y,$$

called evaluation morphism. To define the composition map, one ought to find, for $X, Y, Z \in \mathcal{K}$, a morphism

$$[Y, Z] \otimes [X, Y] \to [X, Z]$$

which under the adjunction corresponds to a morphism

$$([Y, Z] \otimes [X, Y]) \otimes X \to Z.$$

This can be defined as a composition

$$([Y, Z] \otimes [X, Y]) \otimes X \xrightarrow{a} [Y, Z] \otimes ([X, Y] \otimes X) \xrightarrow{1 \otimes \mathrm{ev}} [Y, Z] \otimes Y \xrightarrow{\mathrm{ev}} Z,$$

where the first map is the associator and the second and last are induced by the evaluation morphisms. The reader interested in this approach can consult [Kel82], section 1.6.

Example 3.4. A dg-category $\mathcal{A}$ with only one object X is identifiable the dg-algebra $\mathcal{A}(A, A)$. Recall that a dg-algebra A is a chain complex with a suitable multiplication operation $A \otimes A \to A$. Of course, the composition $\mathcal{A}(A, A) \otimes \mathcal{A}(A, A) \to \mathcal{A}(A, A)$ induces the multiplication operation of the algebra.

On the basis of this example, one can consider a dg-category as a "dg-algebra with several objects", following the example of [Mit72]. This point of view is often very useful, since many constructions typical of dg-algebras and their representations can be translated with little effort in the language of dg-categories.

Example 3.5. If $\mathcal{A}$ is a dg-category, one can define the opposite dg-category $\mathcal{A}^{op}$ with the same objects as $\mathcal{A}$ and hom-spaces $\mathcal{A}^{op}(A, B) = \mathcal{A}(B, A)$ by defining the composition of homogeneous elements $f \in \mathcal{A}^{op}(A, B)^n$ and $g \in \mathcal{A}^{op}(B, C)^n$ as $g \circ_{\mathcal{A}^{op}} f = (-1)^{pq} f \circ_{\mathcal{A}} g$.

Example 3.6. If A is any k-linear category, then we can consider A as a dg-category $\mathcal{A}$ by setting $\mathcal{A}(X,Y)^0 = A(X,Y)$ and $\mathcal{A}(X,Y)^n = 0$ for $n \neq 0$. In this case, we say that $\mathcal{A}$ is concentrated in degree 0. Similarly, is A is a graded category we can consider it as a dg-category with zero differentials.

If $\mathcal{A}$ is a dg-category, one can define several related objects:

- The graded categories $Z^*\mathcal{A}$ and H^*A defined by having the same objects as $\mathcal{A}$ but as a hom-objects the graded modules $Z^*\mathcal{A}(A,B)$ and $H^*A(A,B)$. This is well defined: for any $A,B,C \in \mathcal{A}$, it follows from the graded Leibniz rule that for two homogeneous composable morphisms $f \in \mathcal{A}(B,C)^n$, $g \in \mathcal{A}(A,B)^m$ such that $df = dg = 0$, we have
$$d(fg) = df \circ g + (-1)^n f \circ dg = 0$$
and, if moreover $f = d\varphi$, $g = d\gamma$ for some $\varphi \in \mathcal{A}(B,C)^{n-1}$ and $\gamma \in \mathcal{A}(A,B)^{m-1}$, then
$$f \circ g = d\varphi \circ d\gamma = d(\varphi \circ \gamma).$$
- The k-linear categories $Z^0\mathcal{A}$ and H^0A, again with the same objects but this time with the k-modules $Z^0\mathcal{A}(A,B)$ and $H^0A(A,B)$ as hom-objects. Similarly, one can define the category $\mathcal{A}^0$ having as hom-spaces the k-module $\mathcal{A}^0(A,B)$.

$Z^0\mathcal{A}$ is called the underlying category of $\mathcal{A}$, and $H^0\mathcal{A}$ its homotopy category. Two objects are said to be isomorphic if the are isomorphic in $Z^0\mathcal{A}$, and homotopy equivalent if they are isomorphic in $H^0\mathcal{A}$.

Remark. The notion of isomorphism is the correct one in the sense that if two objects A and A' are isomorphic, then there are natural isomorphisms
$$\mathcal{A}(A,B) \cong \mathcal{A}(A',B) \tag{3.1}$$
and
$$\mathcal{A}(B,A) \cong \mathcal{A}(B,A') \tag{3.2}$$
given by (pre)composition with the isomorphism. The graded Leibniz rule guarantees that those are chain maps.

Example 3.7. In the case of the dg-category $\mathbf{C}_{\mathrm{dg}}(k)$, we have already seen (although using different words) in section 3.1.1 that $Z^0\mathbf{C}_{\mathrm{dg}}(k) = \mathbf{C}(k)$ and that $H^0\mathbf{C}_{\mathrm{dg}}(k) = \mathcal{K}(k)$, the homotopy category of complexes of k-modules.

One has also the notion of a dg-functor.

Definition 3.16. A dg-functor between dg-categories is a $\mathbf{C}(k)$-functor.

That is, a dg functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is composed of

- A map $\mathcal{F} : \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$;
- For every $A, B \in \mathcal{A}$, a chain map $\mathcal{A}(A, B) \to \mathcal{B}(\mathcal{F}A, \mathcal{F}B)$

satisfying the same compositions and identity conditions of Definition 3.3; one defines similarly graded functors between graded categories. As in the case of an ordinary category, we will denote with $\mathcal{F}(f) \in \mathcal{B}(\mathcal{F}A, \mathcal{F}B)$ the image of $f \in \mathcal{A}(A, B)$. A dg-functor is said to be fully faithful if for every $A, B \in \mathcal{A}$ the chain map

$$\mathcal{A}(A, B) \to \mathcal{B}(\mathcal{T}A, \mathcal{T}B)$$

is an isomorphism. A fully faithful dg-functor identifies $\mathcal{A}$ with a full dg-subcategory of $\mathcal{B}$.

Since

$$\mathcal{A}(A, B) \to \mathcal{B}(\mathcal{F}A, \mathcal{F}B)$$

is a chain map, a dg-functor $\mathcal{F}$ induces k-linear functors

$$Z^0\mathcal{F} \colon Z^0\mathcal{A} \to Z^0\mathcal{B}$$

and

$$H^0\mathcal{F} \colon H^0\mathcal{A} \to H^0\mathcal{B}$$

defined as $\mathcal{F}$ on the objects and acting on hom-spaces via the maps induced by the chain maps $\mathcal{A}(A, B) \to \mathcal{B}(\mathcal{F}A, \mathcal{F}B)$; the graded Leibniz rule guarantees that this is well defined. A dg-functor $\mathcal{F}$ is said to be essentially surjective if $Z^0\mathcal{F}$ is essentially surjective. Compositions of dg-functors is defined in the obvious way; similarly, given a dg-category $\mathcal{A}$ we have the identity dg-functor $\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$. In the following, we will sometimes write $\mathcal{F}$ for both $H^0\mathcal{F}$ and $Z^0\mathcal{F}$, letting the meaning be clear from the context.

Definition 3.17. The category $\mathbf{dgcat}_k$ is the category whose objects are small dg-categories and whose morphisms are dg-functors.

$\mathbf{dgcat}_k$ itself is not a dg-category in any natural sense; this is a clear difference from the case of ordinary categories, and one of the main drawbacks of the theory; however, it still possesses several interesting features. For one, as we will now see, it is a 2-category; there exists a notion of transformation between dg-functors.

Definition 3.18. Let $\mathcal{F}, \mathcal{G} \colon \mathcal{A} \to \mathcal{B}$ be dg-functors between two dg-categories. The complex of graded natural transformations $\mathcal{N}at_{\mathrm{dg}}(\mathcal{F}, \mathcal{G})$ is the chain complex having in degree n the module formed by the family of morphisms

$$\varphi_X \in \mathcal{B}(\mathcal{F}X, \mathcal{G}X)^n$$

satisfying a graded naturality condition: for any morphism $f \in \mathcal{A}(X, Y)^m$, we require the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\varphi_X} & \mathcal{G}(X) \\ {\scriptstyle \mathcal{F}(f)}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(Y) & \xrightarrow[\varphi_Y]{} & \mathcal{G}(Y) \end{array}$$

to be commutative up to the sign $(-1)^{nm}$. The differential is defined componentwise.

By definition, $\mathcal{N}at_{\mathrm{dg}}(\mathcal{F}, \mathcal{G})$ is a subcomplex of the product

$$\prod_{X \in \mathcal{A}} \mathcal{B}(\mathcal{F}X, \mathcal{G}X)$$

and inherits from this its differential. In order for this definition to make sense, we ought to prove that if a collection $\varphi_X \in \mathcal{B}(\mathcal{F}X, \mathcal{G}X)^n$ satisfies the naturality condition, then $d\varphi_X \in \mathcal{B}(\mathcal{F}X, \mathcal{G}X)^{n+1}$ does as well. This condition is the reason for the sign in the definition; indeed, let $f \in \mathcal{A}(X, Y)^m$. We want to prove the identity

$$d\varphi_Y \circ \mathcal{F}(f) = (-1)^{(n+1)m} \mathcal{G}(f) \circ d\varphi_X.$$

applying the differential to the naturality condition for φ

$$\varphi_Y \circ \mathcal{F}(f) = (-1)^{nm} \mathcal{G}(f) \circ \varphi_X$$

we get, by the graded Leibniz rule and the fact that $\mathcal{G}$ respects the grading, the identity

$$d\varphi_Y \circ \mathcal{F}(f) + (-1)^n \varphi_Y \circ d\mathcal{F}(f) = (-1)^{nm} d\mathcal{G}(f) \circ \varphi_X + (-1)^{nm+m} \mathcal{G}(f) \circ d\varphi_X,$$

So the claim will follow by proving that

$$(-1)^n \varphi_Y \circ d\mathcal{F}(f) = (-1)^{nm} d\mathcal{G}(f) \circ \varphi_X.$$

But this follows immediately from the fact that $\mathcal{F}$ and $\mathcal{G}$ are chain maps (so $d\mathcal{F}(f) = \mathcal{F}(df)$ and $d\mathcal{G}(f) = \mathcal{G}(df)$) and from the naturality condition applied to the degree $n+1$ morphism df.

Definition 3.19. A dg-natural transformation between two dg functors $\mathcal{F}, \mathcal{G}$ is an element of $Z^0 \mathcal{N}at_{\mathrm{dg}}(\mathcal{F}, \mathcal{G})$.

In this case, we can get rid of the tricky sign and just say that a dg-natural transformation is a collection of closed degree 0 morphisms

$$\varphi_X \in Z^0 \mathcal{B}(\mathcal{F}X, \mathcal{G}X)$$

such that for any for any $f \in \mathcal{A}(X, Y)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\varphi_X} & \mathcal{G}(X) \\ {\scriptstyle \mathcal{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{G}(f)} \\ \mathcal{F}(Y) & \xrightarrow[\varphi_Y]{} & \mathcal{G}(Y) \end{array}$$

commutes.

The above constructions allow us, given two dg-categories $\mathcal{A}$ and $\mathcal{B}$, to define the dg-functor category $\mathcal{H}om(\mathcal{A}, \mathcal{B})$ as a dg-category, with the objects being the dg-functors between $\mathcal{A}$ and $\mathcal{B}$ and, given two dg-functors $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{H}om(\mathcal{A}, \mathcal{B})(\mathcal{F}, \mathcal{G}) = \mathcal{N}at_{\mathrm{dg}}(\mathcal{F}, \mathcal{G})$.

We say that two dg-functors $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if they are isomorphic in $Z^0 \mathcal{H}om(\mathcal{A}, \mathcal{B})$. As in the case of ordinary categories, we have

Proposition 3.20. *Let $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ be a dg-functor between dg-categories. $\mathcal{F}$ is fully faithful and essentially surjective if and only if there exists a dg-functor $\mathcal{G} \colon \mathcal{B} \to \mathcal{A}$ such that $\mathcal{F}\mathcal{G} \cong \mathrm{id}_{\mathcal{B}}$ and $\mathcal{G}\mathcal{F} \cong \mathrm{id}_{\mathcal{A}}$. $\mathcal{G}$ is called a quasi-inverse to $\mathcal{F}$, and is unique up to an isomorphism. A dg-functor admitting a quasi-inverse is called an equivalence of dg-categories*[2].

[2] And not a quasi-equivalence, which is a term that we will use extensively later.

Proof. The proof goes exactly as in the case of the same claim in ordinary category theory, relying crucially on the isomorphisms (3.1) and (3.2). □

Remark. Beware that at this point we have used the notation $\mathcal{H}om(X, Y)$ to denote two different objects; if X and Y are chain complexes, it is the internal hom of chain complexes; if X and Y are dg-categories, it represents the dg-category of dg-functors between X and Y.

The existence of an internal hom might suggest that the category of $\mathbf{dgcat}_k$ might itself have the structure of a closed monoidal category, i.e. it might possess a suitable notion of tensor product of dg-categories: this is in fact the case.

Definition 3.21. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. their tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined as the dg category having as objects the couples (A, B) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, with

$$\mathcal{A} \otimes \mathcal{B}((A, B), (A', B')) = \mathcal{A}(A, A') \otimes \mathcal{B}(B, B')$$

and composition defined on homogeneous elements $f \in \mathcal{A}(A', A''), f' \in \mathcal{A}(A, A'), g \in \mathcal{B}(B', B'')$ and $g' \in \mathcal{B}(B, B')$ as

$$(f \otimes g) \circ (f' \otimes g') = (-1)^{\deg(f)\deg(g)}(f \circ f') \otimes (g \circ g').$$

We do not give the proof of the following proposition, that is analogous to the same claim for the cartesian product of ordinary categories.

Proposition 3.22. *For any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{dgcat}_k$ there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{dgcat}_k}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_{\mathbf{dgcat}_k}(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})).$$

Corollary 3.23. *In the same hypotheses as above there is an isomorphism in $\mathbf{dgcat}_k$*

$$\mathcal{H}om(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathcal{H}om(\mathcal{A}, \mathcal{H}om(\mathcal{B}, \mathcal{C})).$$

Proof. Follows immediately from proposition 3.14. □

This is the first instance of the fact that dg-categories are in general better behaved than triangulated categories (although their connection is maybe not yet apparent to the reader). In the triangulated case, we have seen in Chapter 2 that there can be no triangulated structure on categories of functors, and similarly there is no easy notion of tensor product of triangulated categories. In the setting of dg-categories, both are fairly natural constructions. This is a common phenomenon: triangulated categories work well enough when one has to deal with a single category, but as soon as one wants to relate different categories many constructions become much harder.

Remark. We could have defined dg-categories as categories enriched over complexes of objects in an arbitrary k-linear abelian category, instead of k-modules. It should be clear that no particular difference would arise in the theory.

3.3 Dg-modules

We now wish to talk about *representable* functors, in order to get a dg version of the Yoneda Lemma. However, since the hom-spaces of a dg-category have values in the category $\mathbf{C}_{\mathrm{dg}}(k)$[3], the objects that one ought to consider are (dg) functors from an arbitrary dg-category to $\mathbf{C}_{\mathrm{dg}}(k)$.

[3] Or equivalently $\mathbf{C}(k)$, since the objects are the same.

Lemma 3.24. *Let $\mathcal{A}$ be a dg-category, and let $A \in \mathcal{A}$. Then are defined two dg-functors*

$$\mathcal{A}(A,-)\colon \mathcal{A} \to \mathbf{C}_{\mathrm{dg}}(k)$$
$$\mathcal{A}(-,A)\colon \mathcal{A}^{op} \to \mathbf{C}_{\mathrm{dg}}(k)$$

Proof. We have to prove that what we defined is indeed a dg-functor: we show the case of the functor $\mathcal{A}(A,-)$, the other is similar. The claim that it is a dg-functor amounts to saying that, for any two objects $X, Y \in A$, the map

$$\begin{aligned} \mathcal{A}(X,Y) &\to \mathcal{H}om(\mathcal{A}(A,X), \mathcal{A}(A,Y)) \\ f \quad &\to \qquad \big[g \to f \circ g\big] \end{aligned}$$

is a chain map. It's clear that the map respects the grading, since the composition of a degree n morphism and a degree m morphism is a degree $n+m$ morphism. It also commutes with the differential, since by definition of internal hom of two chain complexes we have, for homogeneous $f \in \mathcal{A}(X,Y)^n$,

$$d\big[g \to f \circ g\big] = \big[g \to d(f \circ g) - (-1)^n f \circ dg\big] = \big[g \to df \circ g\big]$$

by the graded Leibniz rule. This proves that $\mathcal{A}(A,-)$ is a dg-functor. □

Definition 3.25. Let $\mathcal{A}$ be a dg-category. A left dg $\mathcal{A}$-module is a dg-functor

$$L\colon \mathcal{A} \to \mathbf{C}_{\mathrm{dg}}(k).$$

A right dg $\mathcal{A}$-module is a dg-functor

$$R\colon \mathcal{A}^{op} \to \mathbf{C}_{\mathrm{dg}}(k).$$

Sometimes we might drop the prefix dg, and simply talk about right and left $\mathcal{A}$-modules. At a first glance, it might not be clear why these functors should be called modules. However often it is useful to consider dg-modules more as an actual module that as a functor; consider for example the case where $\mathcal{A}$ is a dg-category with a single object X (and is then identified with the dg-algebra $A = \mathcal{A}(X,X)$). Then a left dg $\mathcal{A}$-module M is a chain complex $M(X)$ together with a chain map

$$\mathcal{A}(X,X) \to \mathcal{H}om(M(X), M(X));$$

by the tensor-hom adjunction this is the same thing as a chain map

$$\mathcal{A}(X,X) \otimes M(X) \to M(X).$$

In this case, it should be clear how M can be considered as an object over which $\mathcal{A}$ acts. After developing some theory we will see some further examples of this fact.

Definition 3.26. Let $\mathcal{A}$ be a dg-category. The dg-category of right dg $\mathcal{A}$-modules is defined as the dg-functor category

$$\mathcal{M}od\text{-}\mathcal{A} = \mathcal{H}om(\mathcal{A}, \mathbf{C}_{\mathrm{dg}}(k)).$$

By, definition, a left dg $\mathcal{A}$-module is an element of $\mathcal{M}od\text{-}\mathcal{A}^{op}$. If nothing is specified, by dg $\mathcal{A}$-module we will mean right dg $\mathcal{A}$-module (and so, by dg $\mathcal{A}^{op}$-module a left dg $\mathcal{A}$-module). We will mainly deal with right dg-modules; the reason will be clear once we will have discussed the dg-Yoneda embedding.

Definition 3.27. We define the category $\mathcal{C}(\mathcal{A}) = Z^0\mathcal{M}od\text{-}\mathcal{A}$. Its objects are dg $\mathcal{A}$-modules, and its morphisms are dg-natural transformations between them.

We say that two dg $\mathcal{A}$-modules are isomorphic if they are isomorphic in $\mathcal{C}(\mathcal{A})$.

Similarly, we define the category of graded $\mathcal{A}$-modules $\mathcal{M}od\text{-}\mathcal{A}_{gr}$ as the category of graded functors from $\mathcal{A}^{op}$ seen a graded category to the category of graded k-modules, whose morphisms are graded natural transformations of degree 0. With this definition, the category $\mathcal{M}od\text{-}\mathcal{A}^0$ is a full subcategory of $\mathcal{M}od\text{-}\mathcal{A}_{gr}$, since a dg-functor is in particular a graded functor. Furthermore, a dg-natural transformation $\mathcal{F}\colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$ induces a morphism $\mathcal{F}_{gr}\colon \mathcal{M}od\text{-}\mathcal{A}_{gr} \to \mathcal{M}od\text{-}\mathcal{B}_{gr}$.

Remark. $\mathcal{C}(\mathcal{A})$ is a complete and cocomplete abelian category. Kernels, cokernels and arbitrary (co)limits can be computed object-wise, so this follows from the fact that $\mathbf{C}(k)$ is a complete and cocomplete abelian category. Similarly, $\mathcal{M}od\text{-}\mathcal{A}_{gr}$ is a complete and cocomplete abelian category; on the other hand, the category $\mathcal{M}od\text{-}\mathcal{A}^0$ is not abelian, since for example the kernel of a graded map of chain complexes is not always a subcomplex of the first complex; this is the reason why we introduced the category $\mathcal{M}od\text{-}\mathcal{A}_{gr}$, as a sort of abelian hull for $\mathcal{M}od\text{-}\mathcal{A}^0$.

If M is a dg $\mathcal{A}$-module, we define the shifted module $M[n]$ as the dg-$\mathcal{A}$ module having $M[n](X) = M(X)[n]$ for $X \in \mathcal{A}$, and $M[n](f) = M(f)[n]$ for a morphism f. As in the case of complexes, we have a natural isomorphism

$$\mathcal{N}at_{\mathrm{dg}}(M, N[1]) \cong \mathcal{N}at_{\mathrm{dg}}(M, N)[1]. \tag{3.3}$$

Definition 3.28. Let $\mathcal{A}$ be a dg-category. Then by Lemma 3.24, to an object $A \in \mathcal{A}$ we can associate the right dg $\mathcal{A}$-module $h_A = \mathcal{A}(-, A)$ and the left dg $\mathcal{A}$-module $\tilde{h}_A = \mathcal{A}(A, -)$. We call a right dg $\mathcal{A}$-module isomorphic to one of the form h_A for some $A \in \mathcal{A}$ representable, and a left dg $\mathcal{A}$-module isomorphic to one of the form $\tilde{h}_A$ corepresentable. A dg-module is said to be free if it is isomorphic to an (arbitrary) sum of shifts of representables.

We come to the main result of this section, one of the most useful basic facts about dg-categories.

Theorem 3.29 (dg-Yoneda lemma)**.** *Let $\mathcal{A}$ be a dg-category, M a right dg $\mathcal{A}$-module and $A \in \mathcal{A}$. Then there is a natural isomorphism of complexes*

$$\begin{aligned} \mathcal{N}at_{dg}(h_A, M) &\cong M(X) \\ \varphi \quad &\to \varphi_A(\mathrm{id}_A). \end{aligned} \tag{3.4}$$

If M is a left $\mathcal{A}$-module, there is a natural isomorphism of complexes

$$\begin{aligned} \mathcal{N}at_{dg}(\tilde{h}_A, M) &\cong M(A) \\ \varphi \quad &\to \varphi_A(\mathrm{id}_A). \end{aligned} \tag{3.5}$$

Proof. We prove the second case, the first being similar. The proof goes as in the case of the ordinary Yoneda Lemma: we wish to construct an inverse to the map

$$\varphi \to \varphi_A(\mathrm{id}_A).$$

Recalling that $\tilde{h}_A(A) = \mathcal{A}(A, A)$, this means that for an arbitrary $x \in M(X)^n$, we want to define a graded natural transformation of degree n

$$\eta^x \colon \tilde{h}_A \to M$$

such that $\eta^x_A(\mathrm{id}_A) = x$. Suppose that such a transformation exists, and let $g \in \tilde{h}_A(B)^m = \mathcal{A}(A, B)^m$. Then by graded naturality, the diagram

$$\begin{array}{ccccc}
\tilde{h}_A(A) & & \xrightarrow{\quad\eta^x_A\quad} & & M(A) \\
\Big\downarrow & \mathrm{id}_A & \longrightarrow & x & \Big\downarrow \\
{\scriptstyle \tilde{h}_A(f)} & \Big\downarrow & & \downarrow & {\scriptstyle M(f)} \\
 & \Big\downarrow & & M(f)(x) & \\
 & f & \longrightarrow \ \eta^x_B(f) & & \\
\Big\downarrow & & & & \Big\downarrow \\
\tilde{h}_A(A) & & \xrightarrow[\quad\eta^x_B\quad]{} & & M(B)
\end{array}$$

is commutative up to the sign $(-1)^{nm}$. So the only possible definition of η^x is

$$\begin{aligned} \eta^x_B \colon \tilde{h}_A(B)^m &\to M(B)^{n+m} \\ f \quad &\to (-1)^{nm} M(f)(x). \end{aligned} \tag{3.6}$$

At this point, there are several things to check:

1. That, for any given $B \in \mathcal{A}$, η^x_B is in fact a graded morphism of degree n, i.e. that $M(f)(x) \in M(B)^{n+m}$ for any $f \in \mathcal{A}(A,B)^m$;

2. That (3.6) defines a graded natural transformation of dg-functors;

3. That (3.6) is a two sided inverse to (3.5), i.e. that $\eta^x_A(\mathrm{id}_A) = x$ and $\eta^{\varphi_A(\mathrm{id}_A)} = \varphi$ for an arbitrary graded natural transformation φ;

4. That (3.5) (which at this point will be proved to be bijective) respects the grading and commutes with the differentials, and hence is an isomorphism of complexes.

1. This is immediate: since M is a dg-functor, $M(f)$ is a graded map of degree m (since f has degree m) and so, having x degree n, $M(f)(x) \in M(B)^{n+m}$.

2. In order to prove that η^x is a graded natural transformation of degree n, we have to check for any $B, C \in \mathcal{A}$ and $g \in \mathcal{A}(B,C)^i$, the diagram

$$\begin{array}{ccc} \tilde{h}_A(B) & \xrightarrow{\eta^x_B} & M(B) \\ {\scriptstyle \tilde{h}_A(g)}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{M}(g)} \\ \tilde{h}_A(C) & \xrightarrow{\eta^x_B} & M(C) \end{array}$$

commutes up to the sign $(-1)^{in}$. Writing the definitions and using the fact that M is a dg-functor (so commutes with the compositions) gives

$$\begin{array}{ccccc} \tilde{h}_A(B) & & \xrightarrow{\qquad\qquad\eta^x_B\qquad\qquad} & & M(B) \\ \big\downarrow & f & \longrightarrow & (-1)^{nm}M(f)(x) & \big\downarrow \\ & \big\downarrow & & \big\downarrow & \\ {\scriptstyle \tilde{h}_A(g)}\big\downarrow & & & (-1)^{nm}M(g)M(f)(x) & \big\downarrow{\scriptstyle M(g)} \\ & \big\downarrow & & & \\ \big\downarrow & g \circ f & \longrightarrow & (-1)^{nm+in}M(g \circ f)(x) & \big\downarrow \\ \tilde{h}_A(C) & & \xrightarrow{\qquad\qquad\eta^x_B\qquad\qquad} & & M(C). \end{array}$$

This proves the claim.

3. The first claim is obvious. To prove that the second, we fix a graded natural transformation $\varphi \in \mathcal{N}at_{\mathrm{dg}}(\tilde{h}_A, M)^n$. By definition, for any $B \in \mathcal{A}$ and $f \in \tilde{h}_A(B)^n = \mathcal{A}(A, B)^n$,

$$\eta_B^{\varphi_A(\mathrm{id}_A)}(f) = (-1)^{nm} M(f)(\varphi_A(\mathrm{id}_A))$$

since $\varphi_A(\mathrm{id}_A)$ has degree n. But since φ is a graded natural transformation of degree n,

$$M(f)(\varphi_A(\mathrm{id}_A)) = (-1)^{nm}\varphi_B(f),$$

so $\eta_B^{\varphi_A(\mathrm{id}_A)}(f) = \varphi_B(f)$.

4. That (3.5) respects the grading is obvious. The identity

$$(d\varphi_A)(\mathrm{id}_A) = d(\varphi_A(\mathrm{id}_A))$$

follows from the graded Leibniz rule and the fact that id_A has zero differential, and concludes the proof. □

Corollary 3.30. *For any object A and every right $\mathcal{A}$-module M, the morphism*

$$\begin{aligned} \mathcal{CA}(h_A[-n], M) &\to Z^n M(A) \\ \varphi &\to \varphi_A(\mathrm{id}_A) \end{aligned} \tag{3.7}$$

is an isomorphism of k-modules.

Proof.

$$\begin{aligned} \mathcal{CA}(h_A[-n], M) =& Z^0\mathcal{N}at_{\mathrm{dg}}(h_A[-n], M), \cong Z^0\mathcal{N}at_{\mathrm{dg}}(h_A, M)[n] \cong \\ \cong& Z^n\mathcal{N}at_{\mathrm{dg}}(h_A, M)[n] \cong Z^n M(A). \end{aligned}$$

□

Note that id_A is a closed element of degree n of the chain complex $h_A[-n](A)$.

Corollary 3.31 (dg-Yoneda embedding)**.** *The dg-Yoneda embedding*

$$\begin{aligned} h_{\mathcal{A}} \colon \mathcal{A} &\to \mathcal{M}od\text{-}\mathcal{A} \\ A &\to h_A \end{aligned} \tag{3.8}$$

is a fully faithful dg-functor.

Proof. We have already proved that h_A is a dg-module. We need to prove that (3.8) is a dg-functor, and that it is fully faithful. For the first part, we use the fact that a morphism $f \in \mathcal{A}(A,B)^n$ induces by composition a graded natural transformation of degree n

$$\begin{aligned} h_A &\to h_B \\ g &\to f \circ g, \end{aligned}$$

so (3.8) is a dg-functor. Full faithfulness follows from the dg-Yoneda lemma:

$$\mathcal{M}od\text{-}\mathcal{A}(h_A, h_B) = \mathcal{N}at_{\mathrm{dg}}(h_A, h_B) \cong h_B(A) = \mathcal{A}(A, B).$$

□

This should make it somewhat clear why right dg $\mathcal{A}$-modules are more common than left modules; $\mathcal{A}$ embeds as a full dg-subcategory of the dg-category of right dg $\mathcal{A}$-modules.

A possible interpretation of the dg-Yoneda lemma (but far from the only one) is to consider it as a generalization of the usual isomorphism

$$\mathrm{Hom}_k(k, M) \cong M$$

for a k-module M; in this spirit, we conclude the section with a corollary that will be useful in the future, as well as hopefully give some further intuition about why it is useful to consider a dg-module as an actual module. It is the dg-categorical counterpart of the fact that any module receives a surjective morphism from a free module.

Corollary 3.32. *Let M be a right dg $\mathcal{A}$-module. Then there exists a free (recall Definition 3.28) dg $\mathcal{A}$-module P and a dg-natural transformation $P \to M$ such that for any $n \in \mathbb{Z}$ and for any $A \in \mathcal{A}$, the induced map*

$$Z^n P(A) \to Z^n M(A)$$

is surjective.

The idea of the proof is simple: the representable module h_A, when applied to A itself, possesses a copy of the base ring in degree 0; by taking several copies of h_A and sending[4] the copies of 1 to to the generators of $Z^0 M(A)$ we can find a surjective morphism $h_A(A) \to M(A)$. At this point we can conclude by taking a sum over all the objects of $\mathcal{A}$ of the such formed modules (and shifts thereof.)

[4]This made possible by the dg-Yoneda Lemma.

Proof. First, let $x \in Z^n M(A)$ for some $A \in \mathcal{A}$. Then, by Corollary 3.30 we can find a dg-natural transformation $\eta^x \colon h_A[-n] \to M$ such that $\eta^x_A(\mathrm{id}_A) = x$. At this point, we find a collections of generators $\{x_i\}_{i \in I}$ of $Z^n M(A)$ and define the dg-natural transformation

$$\Lambda^{A,n} \colon \bigoplus_I h_A[-n] \to M$$

by applying η^{x_i} to the i-th addendum. This, way, we get that

$$\Lambda^{A,n}_A \colon \big(\bigoplus_I h_A[-n]\big)(A) \to M(A)$$

is surjective on the n-cycles. At this point, it is enough to repeat the process for all the objects of $\mathcal{A}$ and all $n \in \mathbb{Z}$; the desired dg-natural transformation

$$\Lambda : \bigoplus_{\substack{A \in \mathcal{A} \\ n \in \mathbb{Z}}} \bigoplus_I h_A[-n] \to M$$

can be defined by applying $\Lambda^{A,n}$ to the addendum (of the first sum) indexed by (A, n). □

The reason why we were only able to achieve the surjectivity on the level of the cycles is the fact that we only relied on the images of the identities, that are themselves closed. To obtain the general surjectivity we will need to consider objects slightly more complicated than free modules. This will be done in section 3.5.1.

3.3.1 Tensor product of dg-modules

In this section, $\mathcal{A}$ and $\mathcal{B}$ are fixed small dg-categories. The goal of this section is to define the notion of tensor product of two dg $\mathcal{A}$-modules. This will allow us, given a dg-functor $\mathcal{A} \to \mathcal{B}$, to define his extension $\mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$.

Definition 3.33. Let $\mathcal{A}$ and $\mathcal{B}$ be dg-categories. An $\mathcal{A}$-$\mathcal{B}$-bimodule is an $\mathcal{A}^{op} \otimes \mathcal{B}$-module.

Concretely, if we have an $\mathcal{A}$-$\mathcal{B}$-bimodule X, for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have a chain complex $X(A, B)$ and a morphism of complexes

$$\mathcal{A}(A', A) \otimes X(A, B) \otimes \mathcal{B}(B, B') \to X(A', B')$$

for all $A' \in \mathcal{A}$ and $B' \in \mathcal{B}$.

Remark. If X is an $\mathcal{A}$-$\mathcal{B}$-bimodule then, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the dg-functor $X(A, -)$ is a right $\mathcal{B}$-module and $X(-, B)$ a left $\mathcal{A}$-module.

Example 3.8. If $\mathcal{A}$ is a dg-category, then $\mathcal{A}(-,-)$ is an $\mathcal{A}$-$\mathcal{A}$ bimodule. Similarly, if $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a dg-functor, then

$$\mathcal{B}(-, \mathcal{F}(-))$$

is an $\mathcal{A}$-$\mathcal{B}$ bimodule.

Definition 3.34. Let $M \in \mathcal{M}od\text{-}\mathcal{A}$ be a right dg $\mathcal{A}$-module and $N \in \mathcal{M}od\text{-}\mathcal{A}^{op}$ a left dg $\mathcal{A}$-module. Define the chain complex T as

$$T = \bigoplus_{A \in \mathcal{A}} M(A) \otimes_k N(A),$$

where we have denoted with $\otimes_k$ the tensor product of chain complexes of k-modules. Let $v \in M(B)$, $f \in \mathcal{A}(A, B)$ and $w \in N(A)$ for some $A, B \in \mathcal{A}$. Define the subcomplex $S \subseteq T$ generated by the elements

$$M(f)v \otimes w - v \otimes N(f)w.$$

The tensor product $M \otimes_{\mathcal{A}} N$ is defined as the chain complex T/S.

Another say of saying this is that the tensor product $M \otimes_{\mathcal{A}} N$ is defined as the quotient of the chain complex T by the relations

$$M(f)v \otimes w = v \otimes N(f)w \tag{3.9}$$

for f, v and w as in the definition.

Remark. The construction is functorial: a graded transformation $\varphi \in \mathcal{N}at_{\mathrm{dg}}(M, M')^n$ induces a graded map of degree n

$$\varphi \otimes \mathrm{id}_N\colon M \otimes N \to M' \otimes N$$

defined as

$$\varphi \otimes \mathrm{id}_N(v \otimes w) = (-1)^{nm} \varphi_A(v) \otimes w$$

for $v \in M(A)^m, w \in N(A)$. The fact that φ is graded natural transformation guarantees that this is well defined: for $f \in \mathcal{A}(A, A')^m$, $v \in M(A')^l$ and $w \in N(A)$,

$$\begin{aligned}&\varphi \otimes \mathrm{id}_N(M(f)v \otimes w) = (-1)^{n(l+m)}(\varphi_A \circ M(f)v) \otimes w = \\ =&(-1)^{nl}(M(f) \circ \varphi_A v) \otimes w = (-1)^{nl} \varphi_A v \otimes N(f)w = \varphi \otimes \mathrm{id}_N(v \otimes N(f)w).\end{aligned}$$

Example 3.9. If X is an $\mathcal{A}$-$\mathcal{B}$ bimodule and M an $\mathcal{A}$ module, then $M \otimes_{\mathcal{A}} X$ is a $\mathcal{B}$-module. Explicitly, it is the $\mathcal{B}$-module assigning the complex $M \otimes_{\mathcal{A}} X(-, B)$ to the object B. The above remark implies that this defines a dg- functor

$$\begin{aligned} \mathcal{M}od\text{-}\mathcal{A} &\to \mathcal{M}od\text{-}\mathcal{B} \\ M &\to M \otimes_{\mathcal{A}} X \end{aligned} \tag{3.10}$$

We now want to prove that the functor given by tensoring by a bimodule is the right adjoint to an appropriately defined hom functor.

Definition 3.35. Let X be an $\mathcal{A}$-$\mathcal{B}$ bimodule and N be a right dg $\mathcal{B}$-module. The right dg $\mathcal{A}$-module $\mathcal{H}om(X, N)$ is defined as

$$A \to \mathcal{N}at_{\mathrm{dg}}(X(A, -), N).$$

For any object $A \in \mathcal{A}$. This defines a natural dg-functor

$$\begin{aligned} \mathcal{M}od\text{-}\mathcal{B} &\to \mathcal{M}od\text{-}\mathcal{A} \\ N &\to \mathcal{H}om(X, N). \end{aligned}$$

Proposition 3.36. *The functors $- \otimes_{\mathcal{A}} X$ and $\mathcal{H}om(X, -)$ form an adjoint pair at the level of the underlying categories. Explicitly, for any $\mathcal{A}$-module M and $\mathcal{B}$-module N there exists a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M, \mathcal{H}om(X, N)) \cong \mathrm{Hom}_{\mathcal{C}(\mathcal{B})}(M \otimes_{\mathcal{A}} X, N).$$

This proof is a long unwinding of the definitions, relying essentially on the tensor-hom adjunction of chain complexes.

Proof. An element of $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(M, \mathcal{H}om(X, N))$ is a dg-natural transformation between the two dg $\mathcal{B}$-modules M and $\mathcal{H}om(X, N)$. This means that it is represented by a collection of chain maps

$$\varphi_A \colon M(A) \to \mathcal{N}at_{\mathrm{dg}}(X(A, -), N)$$

satisfying the naturality condition

$$\varphi_A \circ M(f) \cong \mathcal{H}om(X, N)(f) \circ \varphi_{A'}. \tag{3.11}$$

Since $\mathcal{N}at_{\mathrm{dg}}(X(A, -), N)$ is a subcomplex of the product

$$\prod_{B \in \mathcal{B}} \mathcal{H}om(X(A, B), N(B))$$

the chain map φ_A can be expressed as a collection of chain maps

$$\varphi_A^B\colon M(A) \to \mathcal{H}om(X(A,B), N(B))$$

satisfying the graded naturality condition in the variable B

$$\varphi_A^{B'}(x) \circ X(A,g) = (-1)^{nm} N(g) \circ \varphi_A^B(x) \tag{3.12}$$

for $x \in M(A)^n$ and $g \in \mathcal{B}(B,B')^m$. Now, by definition

$$(\mathcal{H}om(X,N)(f) \circ \varphi_{A'}^B)(v) = \varphi_{A'}^B(v) \circ X(f,B)$$

for $v \in M(A')$ and $f \in \mathcal{A}(A,A')$, so condition (3.11) reads

$$\varphi_A^B(M(f)y) = \varphi_{A'}^B(y) \circ X(f,B). \tag{3.13}$$

On the other hand, an element γ of $\mathrm{Hom}_{\mathcal{C}(\mathcal{B})}(M \otimes_{\mathcal{A}} X, N)$ is a collection of chain maps

$$\gamma_B\colon M \otimes_{\mathcal{A}} X(-,B) \to N(B)$$

satisfying the naturality condition in the variable B

$$\gamma_{B'} \circ N(f) = \gamma_B \circ \mathrm{id}_M \otimes X(-,f). \tag{3.14}$$

In turn, since $M \otimes_{\mathcal{A}} X(-,B)$ is a quotient of the chain complex

$$\bigoplus_{A \in \mathcal{A}} M(A) \otimes_k X(A,B).$$

every γ_B can be represented as a collection of chain maps

$$\gamma_B^A\colon M(A) \otimes_k X(A,B) \to N(B)$$

vanishing on the quotienting subcomplex, so such that

$$\gamma_B^A(M(f)v \otimes w) = \gamma_B^{A'}(v \otimes X(f,B)w) \tag{3.15}$$

for $f \in \mathcal{A}(A,A')$, $v \in M(A')$ and $w \in X(A,B)$. We can rewrite condition (3.14) as

$$N(g) \otimes \gamma_{B'}(x \otimes s) = (-1)^{nm} \gamma_B(x \otimes X(A,g)s) \tag{3.16}$$

for $x \in M(A)^n$, $g \in \mathcal{B}(B,B')^m$ and $s \in X(A,B')$. At this point, it only remains to check that under the isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{C}(k)}(M(A) \otimes_k X(A,B), N(B)) & \cong & \mathrm{Hom}_{\mathbf{C}(k)}(M(A), \mathcal{H}om(X(A,B), N(B))) \\ f & \to & x \to [y \to f(x \otimes y)] \end{array}$$

the conditions imposed on γ_B^A correspond to those on φ_A^B. However, it is now clear that the condition (3.13) is equivalent to (3.15) and condition (3.12) is equivalent to (3.16). □

One could have defined the tensor product directly using the adjunction of Proposition 3.36. Concretely, one would define the (easier to deal with) functor $\mathcal{H}om(X, -)$ and then prove abstractly that it admits a left adjoint. Another possible (and maybe more abstractly sound than the one we gave) approach is via Kan extensions and coends, see for example [Gen17] or [Dri04] 14.3.

Corollary 3.37. *Let $A \in \mathcal{A}$, $M \in \mathcal{M}od\text{-}\mathcal{A}$ and $N \in \mathcal{M}od\text{-}\mathcal{A}^{op}$. Then there are natural isomorphisms*

$$h_A \otimes_{\mathcal{A}} N \cong M(A)$$

and

$$M \otimes_{\mathcal{A}} \tilde{h}_A \cong M(A).$$

Proof. We prove the first claim, the second being similar. Denote with $\mathcal{K}$ the dg-category with one object X and $\mathcal{K}(X, X) = k$, with k considered as a chain complex concentrated in degree 0. Then a dg $\mathcal{K}$-module is the same thing as a chain complex, and a dg $\mathcal{A}^{op}$-module can be seen as an $\mathcal{A}$-$\mathcal{K}$ bimodule. Since $\mathcal{C}(\mathcal{K}) = \mathbf{C}(k)$, Proposition 3.36 gives, for an arbitrary chain complex C and considering N as an $\mathcal{A}$-$\mathcal{K}$ bimodule, an isomorphism

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{K})}(h_A \otimes_{\mathcal{A}} N, C) \cong \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(h_A, \mathcal{H}om(N, C)).$$

By the dg-Yoneda lemma,

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(h_A, \mathrm{Hom}_{\mathcal{C}(\mathcal{K})}(N, C)) \cong \mathrm{Hom}_{\mathcal{C}(\mathcal{K})}(N, C)(A) = \mathcal{H}om(N(A), C) = \\ &= \mathrm{Hom}_{\mathbf{C}(k)}(N(A), C).\end{aligned}$$

So

$$\mathrm{Hom}_{\mathbf{C}(k)}(h_A \otimes_{\mathcal{A}} N, C) \cong \mathrm{Hom}_{\mathbf{C}(k)}(N(A), C),$$

and the claim follows from Yoneda's lemma. □

Corollary 3.38. *In the same hypotheses as Proposition 3.36,*

$$\mathcal{N}at_{dg}(M, \mathcal{H}om(X, N)) \cong \mathcal{N}at_{dg}(M \otimes_{\mathcal{A}} X, N).$$

Proof. This is analogous to Proposition 3.14. □

Definition 3.39. Let $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. The restriction dg-functor is defined as

$$\begin{aligned}\mathrm{Res}_{\mathcal{F}} \colon \mathcal{M}od\text{-}\mathcal{B} &\to \mathcal{M}od\text{-}\mathcal{A} \\ N &\to N \circ F\end{aligned}$$

The induction dg-functor is defined as

$$\begin{aligned}\operatorname{Ind}_{\mathcal{F}}\colon \mathcal{M}od\text{-}\mathcal{A} &\to \mathcal{M}od\text{-}\mathcal{B}\\ M &\to M \otimes_{\mathcal{A}} \mathcal{B}(-,\mathcal{F}(-))\end{aligned}$$

The restriction and induction functors can be considered as generalizations of the classical restriction and induction functors in the case of a morphism of rings. In analogy with this case some texts denote $M\otimes_{\mathcal{A}}\mathcal{B}(-,\mathcal{F}(-))$ with $M \otimes_{\mathcal{A}} \mathcal{B}$. As in the classical case, we have

Proposition 3.40. *$\operatorname{Res}_{\mathcal{F}}$ and $\operatorname{Ind}_{\mathcal{F}}$ form an adjoint pair between the underlying categories, with the restriction being the right adjoint and the induction being the left adjoint.*

Proof. This follows from Proposition 3.36 once we prove that

$$\mathcal{H}om(\mathcal{B}(-,\mathcal{F}(-))M) \cong \operatorname{Res}_{\mathcal{F}} M.$$

Indeed, recall that the dg $\mathcal{A}$-module $\operatorname{Res}_{\mathcal{F}} M$ is defined by

$$A \to M(\mathcal{F}A)$$

For $A \in \mathcal{A}$. On the other hand, the dg $\mathcal{A}$-module $\mathcal{H}om(\mathcal{B}(-,\mathcal{F}(-)),M)$ is defined by

$$A \to \mathcal{N}at_{\mathrm{dg}}(\mathcal{B}(-,\mathcal{F}(A)),M) \cong M(\mathcal{F}A)$$

by the dg-Yoneda lemma.

□

Another crucial property of $\operatorname{Ind}_{\mathcal{F}}$ is that it acts as an extension of $\mathcal{F}$: when when restricted to $\mathcal{A} \subseteq \mathcal{M}od\text{-}\mathcal{A}$, it coincides with $\mathcal{F}$.

Proposition 3.41. *Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. Then the diagram*

$$\begin{array}{ccc}\mathcal{A} & \xrightarrow{h_{\mathcal{A}}} & \mathcal{M}od\text{-}\mathcal{A}\\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \operatorname{Ind}_{\mathcal{F}}}\\ \mathcal{B} & \xrightarrow{h_{\mathcal{B}}} & \mathcal{M}od\text{-}\mathcal{B}\end{array}$$

commutes up to isomorphism in $\mathcal{M}od\text{-}\mathcal{B}$..

Proof. For an arbitrary $A \in \mathcal{A}$,

$$\operatorname{Ind}_{\mathcal{F}} h_A = h_A \otimes_{\mathcal{A}} \mathcal{B}(-,\mathcal{F}(-)) \cong \mathcal{B}(-,\mathcal{F}(A)) = h_{\mathcal{F}(A)}$$

by Corollary 3.37. □

3.4 Triangulated properties

In this section, we will prove that the category of dg $\mathcal{A}$-modules up to homotopy is in a natural way a triangulated category. This, since any category embeds in its category of modules, will allow us to define the notion of a pretriangulated dg-category, a dg-category whose homotopy category is triangulated. We will then see that an arbitrary triangulated category has a canonical pretriangulated hull.

Definition 3.42. The homotopy category $\mathcal{HA}$ of dg $\mathcal{A}$-modules is the k-linear category

$$\mathcal{HA} = H^0\mathcal{M}od\text{-}\mathcal{A}.$$

Two dg $\mathcal{A}$-modules are said to be homotopy equivalent if they are isomorphic in $\mathcal{HA}$.

Remark. By the dg-Yoneda Lemma, there is a natural isomorphism

$$\mathcal{HA}(h_A[-n], M) \cong H^n M(A),$$

so the dg-Yoneda embedding induces a fully faithful functor

$$H^0 h_{\mathcal{A}} \colon H^0\mathcal{A} \to \mathcal{HA}.$$

We call $H^0 h_{\mathcal{A}}$ the derived Yoneda embedding. When there is no risk of ambiguity, we will use simply $h_{\mathcal{A}}$ to denote the derived Yoneda embedding.

The category $\mathcal{HA}$ is clearly preadditive, since its hom-spaces have a natural additive structure. The natural functor $\mathcal{CA} \to \mathcal{HA}$ preserves finite coproducts: indeed,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{HA}}(\bigoplus_i X_i, Y) = H^0\mathcal{N}at_{\mathrm{dg}}(\bigoplus_i X_i, Y) \cong H^0 \prod_i \mathcal{N}at_{\mathrm{dg}}(X_i, Y) \cong \\ \cong \prod_i H^0\mathcal{N}at_{\mathrm{dg}}(X_i, Y) \cong \prod_i \mathrm{Hom}_{\mathcal{HA}}(X_i, Y)\end{aligned}$$

so coproducts in $\mathcal{CA}$ are in particular coproducts in $\mathcal{HA}$. Therefore, $\mathcal{HA}$ is an additive category with small coproducts. The category $\mathcal{HA}$ looks very similar to the homotopy category $\mathcal{K}(k)$ of chain complexes of k-modules.; in fact, in the case $\mathcal{A} = \mathcal{K}$, they coincide[5]. For this reason, it makes sense to expect that, like $\mathcal{K}(k)$, $\mathcal{HA}$ admits a triangulated structure. There are several ways to do this. One, taken originally in [Kel94], is to prove that $\mathcal{C}(\mathcal{A})$ is in a natural way a Frobenius category, and then to show that $\mathcal{HA}$ identifies with its stable category, and is hence triangulated. Another, conceptually not very

[5] The choice of notation is somewhat unfortunate, but one should not confuse the dg-category $\mathcal{K}$ with the homotopy category of complexes of k-modules $\mathcal{K}(k)$.

different, is to repeat *verbatim* the proof that the homotopy category $\mathcal{K}(\mathbf{A})$ of an abelian category $\mathbf{A}$ is triangulated and observe that the arguments translate effortlessly in the case of the homotopy category of dg-modules. Here we will sketch the second approach; details on the first can be found in [Kel94].

In all this section, $\mathcal{A}$ will be a fixed dg-category.

Definition 3.43. Let $f\colon M \to N$ be a closed, degree 0 morphism between dg $\mathcal{A}$-modules.[6] Its (strict) cone $C(f)$ is the dg $\mathcal{A}$-module defined in the following way:

[6] That is, a morphism in $\mathcal{C}(\mathcal{A})$.

- For an object $A \in \mathcal{A}$, set

$$C(f)(A) = C(f_A) = M(A)[1] \oplus N(A)$$

  as a graded module, with the differential defined as

$$d(x^{n+1}, y^n) = (-d_A x^{n+1}, d_{N(A)} y^n + f_A(x^{n+1}))$$

  for $x^{n+1} \in M(A)[1]^n = M(A)^{n+1}$, $y^n \in N(A)^n$. It is immediate to verify that $d^2 = 0$.

- Given a morphism $h \in \mathcal{A}(B, A)^i$, we define the morphism

$$C(f)(h)\colon C(f)(A) \to C(f)(B)$$

  as

$$C(f)(h)(x^{n+1}, y^n) = ((-1)^i M(h) x^{n+1}, N(h) y^n)$$

  for $x^{n+1} \in M(B)[1]^n$, $y^n \in N(B)^n$.

We verify that

$$C(f)\colon \mathcal{A}^{op}(A, B) \to \mathcal{H}om(C(f)(A), C(f)(B))$$

is a chain map, so to check that $C(f)$ is indeed a dg $\mathcal{A}$-module: we have

$$C(f)(dh)(x^{n+1}, y^n) = ((-1)^{i+1} M(dh) x^{n+1}, N(dh) y^n)$$

while

$$\begin{aligned}
&(dC(f)(h))(x^{n+1}, y^n) = d_{C(f)(B)}(C(f)(h)(x^{n+1}, y^n)) - (-1)^i C(f)(h) d(x^{n+1}, y^n) = \\
&= d_{C(f)(B)}((-1)^i M(h) x^{n+1}, N(h) y^n) - (-1)^i C(f)(h)(-dx^{n+1}, dy^n + f_A(x^{n+1})) = \\
&= ((-1)^{i+1} d(M(h) x^{n+1}), d(N(h) x^{n+1}) + (-1)^i f_B(M(h) x^{n+1})) + \\
&\quad - (-M(h) dx^{n+1}, (-1)^i N(h) dy^n + (-1)^i N(h) f_A(x^{n+1}))) = \\
&= ((-1)^{i+1} M(dh) x^{n+1}, N(dh) y^n)
\end{aligned}$$

by the naturality of f.

The inclusion map $N(A) \to C(f)(A)$ is a chain map and hence defines a dg-natural transformation $N \to C(f)$, while the inclusion $M(A)[1] \to C(f)(A)$ is only a graded map of degree 0 (it does not preserve the differential). On the other hand, the projection $C(f)(A) \to M(A)[1]$ is a chain map, while the projection $C(f)(A) \to N(A)$ is not. So, if $f\colon M \to N$ is a closed degree 0 morphism, we have an exact sequence in $\mathcal{C}(\mathcal{A})$

$$0 \to N \to C(f) \to M[1] \to 0 \tag{3.17}$$

when the first arrow is the inclusion and the second the projection.

Remark. As in the case of complexes, the cone is a homotopical cokernel, in the sense that the datum of a morphism from the cone of a morphism $f\colon M \to N$ to a dg $\mathcal{A}$-module L is the defines univocally a morphism g from N to S, together with a nullhomotopy of gf. One of the fundamental facts about triangulated categories is that homotopical cokernels are also homotopical kernels (see Proposition 3.61, as well as axiom TR2). This echoes the fact that, in general, triangulated categories arise when taking homotopy categories of (appropriately defined) "stable" categories.

Remark. From now on, we will often make the abuse of notation of talking about dg-modules as if they were actual chain complexes: for example, by talking about an element $x \in M^n$ if M is a dg $\mathcal{A}$-module. Every time we write this, it will be implied that we are actually talking about an element $x_A \in M(A)^n$ for an arbitrary A. This will not create problems since all the constructions we will make will be natural in A, so the "object-less" constructions we will make will give well defined dg-modules and dg-natural transformations. Of course, this is not obvious *a priori*, so one should check every time that a definition given talking the object-free language actually translates into a meaningful construction in the many-objects setting, like we just did for the cone of a dg-natural transformation.

Definition 3.44. A short exact sequence of dg $\mathcal{A}$-modules

$$0 \to M \xrightarrow{u} N \xrightarrow{v} L \to 0$$

is said to be graded split (semi-split in [GM02]) if there exists an element $w \in \mathcal{M}od\text{-}\mathcal{A}(L, N)^0$ such that $v \circ w = \mathrm{id}_Z$.

We are not requiring w to commute with the differential, but only to respect the grading. This implies (by the usual splitting lemma in the abelian category $\mathcal{M}od\text{-}\mathcal{A}_{gr}$) that N admits a decomposition $N \cong M \oplus L$ as a graded module. The prototypical example of a graded split sequence is the sequence

(3.17), where the middle objects is isomorphic to the sum of the other two as a graded module, but not as a dg-module. In fact, up to homotopy (and up to a rotation of the triangle) this example is completely general: any sequence of the type is (3.17) is graded split (take as w the inclusion); we will see in Lemma 3.46 that the converse is also true. We now get to the main theorem of this section.

Theorem 3.45. *$\mathcal{HA}$ admits the structure of a triangulated category, with the translation given by the shift and the class of distinguished triangles being given by the triangles isomorphic to those of the form*

$$M \xrightarrow{f} N \to C(f) \to M[1].$$

for some closed degree 0 *morphism f. Furthermore, the distinguished triangles are precisely those isomorphic to graded split short exact sequences (see Lemma 3.46).*

The proof is very long, so we give the main steps and refer to [GM02, p. IV.1.9] for the full details. Again, [GM02] deals with chain complexes rather than with dg $\mathcal{A}$-modules, but, as will be clear, the proof is natural in the objects and translates *verbatim* to the dg-case, modulo the necessary checks.

To begin with, recall that exactly as in the case of chain complexes, two morphisms f, g in $\mathcal{C}(\mathcal{A})$ are identified in $\mathcal{HA}$ if and only if there exists a graded natural transformation h of degree -1 such that $d(h) = f - g$.

Lemma 3.46. *Any graded split exact sequence in $\mathcal{HA}$*

$$0 \to M \xrightarrow{u} N \xrightarrow{v} L \to 0$$

can be completed to a distinguished triangle (as in Theorem 1.9)

$$M \to N \to L \to M[1].$$

Proof. Selecting a splitting, we can suppose $N \cong M \oplus L$ as a graded module. The differential of N is easy to identify; indeed, writing the generic degree 1 graded morphism as

$$d_N(x, y) = (\alpha(x) + \beta(y), \gamma(x) + \delta(y))$$

and imposing for $M \to N$ and $N \to L$ to commute with the differentials, we find

$$\alpha(x) = dx, \quad \gamma(x) = 0 \quad \text{and} \quad \beta(y) = dy$$

so $d_N(x,y) = (dx - fy, dy)$ for some $f \in \mathcal{N}at_{\mathrm{dg}}(L,M)^1$. In this form, the condition $d^2 = 0$ is equivalent to requiring $df = 0$, so for f to be a dg-natural transformation $L \to X[1]$.

At this point, one proves that the triangle

$$M \xrightarrow{u} N \xrightarrow{v} L \xrightarrow{f} M[1]$$

is distinguished by proving that it is isomorphic to the triangle

$$M \xrightarrow{u} N \to C(u) \to M[1];$$

with this goal, consider the diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{u} & N & \xrightarrow{v} & L & \xrightarrow{f} & M[1] \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
M & \xrightarrow[u]{} & N & \longrightarrow & N \oplus M[1] & \longrightarrow & M[1]
\end{array}$$

with $g(x,y) = (0, \mathrm{id}_L, f)$. One must then prove that the squares commute modulo homotopy (in fact, the first and last square commute in $\mathcal{C}(\mathcal{A})$) and that g is a homotopy equivalence; this concludes the proof. $\square$

We can now approach the proof of theorem 1.9.

Proof of Theorem 1.9. We have already seen that $\mathcal{HA}$ is an additive category, so we have left to prove axioms TR1-TR4.

TR1

b) and c) are obvious from the definition. To prove a), one shows that the diagram

$$\begin{array}{ccccccc}
M & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & M[1] \\
\| & & \| & & \downarrow & & \| \\
M & \longrightarrow & M & \longrightarrow & C(\mathrm{id}_M) & \longrightarrow & M[1]
\end{array}$$

commutes up to homotopy and that $0 \to C(\mathrm{id}_M)$ is a homotopy equivalence. This can be done by finding a graded natural transformation $h\colon C(\mathrm{id}_M) \to C(\mathrm{id}_M)$ of degree -1, such that $d(h) = \mathrm{id}_{C(\mathrm{id}_M)}$. Since $C(\mathrm{id}_M) = M[1] \oplus M$, by defining

$$h(x^{n+1}, y^n) = (y^n, 0)$$

one can easily check that h is a graded natural transformation of degree -1, and that $d(h) = \mathrm{id}_{C(\mathrm{id}_M)}$.

TR2

We wish to prove that if

$$M \xrightarrow{u} N \xrightarrow{v} L \to wM[1]$$

is distinguished then

$$N \xrightarrow{v} L \xrightarrow{w} M[1] \xrightarrow{-u[1]} N[1]$$

is as well. For this, it is enough to prove that, for any morphism u, denoting with i and p the inclusion to and projection from its cone, the triangle

$$N \xrightarrow{i} C(u) \xrightarrow{p} M[1] \xrightarrow{-u[1]} N[1]$$

is isomorphic to the (distinguished by definition) triangle

$$N \xrightarrow{i} C(u) \to C(i) \to N[1].$$

Then, it is enough to define a morphism $\theta\colon M[1] \to C(i)$ such that the diagram

$$\begin{array}{ccccccc} N & \xrightarrow{i} & C(u) & \xrightarrow{p} & M[1] & \xrightarrow{-u[1]} & N[1] \\ \| & & \| & & \downarrow{\theta} & & \| \\ N & \xrightarrow{i} & C(u) & \longrightarrow & C(i) & \longrightarrow & N[1] \end{array} \tag{3.18}$$

is an isomorphism of triangles. Recalling that (as a graded $\mathcal{A}$-module)

$$C(i) = N[1] \oplus C(u) = N[1] \oplus M[1] \oplus N,$$

we can define

$$\theta(x) = (-u[1](x), x, 0).$$

At this point, one has to verify the following:

1. θ is a dg-natural transformation (that is, is a degree 0 natural transformation that commutes with the differentials);

2. The diagram (3.18) commutes up to homotopy;

3. θ is a homotopy equivalence.

This proves axiom TR2.

TR3

Again, we can prove this in the case where both distinguished triangles are in the form $M \xrightarrow{u} N \to C(u) \to M[1]$. In this case, suppose that we have a diagram

$$\begin{array}{ccccccc} M & \xrightarrow{u} & N & \longrightarrow & C(u) & \longrightarrow & M[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\,\vdots & & \downarrow{\scriptstyle f[1]} \\ M' & \xrightarrow{u'} & N' & \longrightarrow & C(u') & \longrightarrow & M'[1] \end{array}$$

with the morphism h missing, commutative up to homotopy. This means that there exists a morphism $s \in \mathcal{N}at_{\mathrm{dg}}(M, N')^{-1}$ such that $ds = g \circ u - u' \circ f$. Recalling that $C(u) = N \oplus M[1]$ and $C(u') = N' \oplus M'[1]$, we can now define $h\colon C(u) \to C(u')$ as

$$h(x, y) = (f(x), g(y) + s(x)).$$

At this point, one verifies that h is natural, commutes with the differentials and that makes the diagram above commute.

Important remark. The morphism h depends explicitly on the chosen homotopy s, which is in general not unique; this echoes the discussion in Chapter 2 about the non-functoriality of the cone in an abstract triangulated category. □

TR4

This is omitted: the interested reader can consult [GM02] for a proof of this as an application of Lemma 3.46. □

Recall now that in a triangulated category the (non-strict) cone of a morphism $x \xrightarrow{u} y$ is any object completing $x \xrightarrow{u} y$ to a distinguished triangle (as per TR1), and that any two cones are (non-canonically) isomorphic.

Lemma 3.47. *Let $\mathcal{T}$ be a triangulated category and suppose that $\mathcal{S} \subseteq \mathcal{T}$ is a full subcategory. Suppose also that the following hold:*

1. $0 \in \mathcal{S}$;
2. *For every $x \in \mathcal{S}$ the translation $x[1]$ is isomorphic to an object in $\mathcal{S}$;*
3. *For every morphism $x \xrightarrow{u} y$ between objects of $\mathcal{S}$, the cone $C(u)$ is isomorphic to an object in $\mathcal{S}$.*

Then $\mathcal{S}$ admits the structure of a triangulated category, with translation functor induced from that of $\mathcal{T}$ (via condition 2) where the distinguished triangles are those that are distinguished in $\mathcal{T}$.

Note that, even though the object $C(u)$ is only defined up to isomorphism, the condition is still well-posed.

Proof. Axioms TR1-TR4 are trivially satisfied, since they only depend on objects of which conditions 1-3 guarantee the existence. Then, since $\mathcal{S}$ has a zero object and its hom-sets form an abelian group, it is also closed under finite coproducts. □

We have just proved that, although the homotopy category of dg $\mathcal{A}$-modules $H^0\mathcal{A}$ is not necessarily triangulated, it always embeds in a triangulated category. In particular, there exist notions of shifts of objects and cones of morphisms; these objects, however, may not lie in $\mathcal{A}$. For this reason, we give the following key definition:

Definition 3.48. A dg-category $\mathcal{A}$ is said to be pretriangulated (resp. strongly pretriangulated) if the following conditions are satisfied:

- $0 \in \mathcal{A}$;
- For every closed degree 0 morphism, its (strict) cone is isomorphic (resp. homotopy equivalent) to an object of $\mathcal{A}$;
- For every object $A \in \mathcal{A}$ its shift $A[1]$ is isomorphic (resp. homotopy equivalent) to an object of A.

Equivalently, $\mathcal{A}$ is pretriangulated (resp. strongly pretriangulated) if the essential image of the derived Yoneda embedding $H^0 h_{\mathcal{A}}$ (resp. the Yoneda embedding $h_{\mathcal{A}}$) is a triangulated subcategory of $\mathcal{H}\mathcal{A}$.

A strongly pretriangulated dg-category is in particular triangulated. If $\mathcal{A}$ is pretriangulated, we will freely talk about shifts and cones of objects.

Proposition 3.49. *Let $\mathcal{A}$ be a pretriangulated dg-category. Then its homotopy category $H^0\mathcal{A}$ admits a triangulated structure, with shift functor and distinguished triangles induced by those of $\mathcal{H}\mathcal{A}$.*

Proof. Follows immediately from Lemma 3.47 and from the definition of pretriangulated dg-category. □

Remark. If $\mathcal{A}$ is a strongly pretriangulated dg-category (in fact, just if it contains shifts), there is a natural isomorphism

$$\mathcal{A}(A, B)^i \cong \mathcal{A}(A[-i], B) \cong \mathcal{A}(A, B[i]).$$

If $\mathcal{A}$ is (not strongly) pretriangulated, there are isomorphisms

$$H^i\mathcal{A}(A, B) \cong H^0\mathcal{A}(A[-i], B) \cong H^0\mathcal{A}(A, B[i]).$$

Lemma 3.50. *Let $\mathcal{F}$ be a dg-functor $\mathcal{F}\colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$. Then the following claims hold:*

1. *$\mathcal{F}$ commutes with shifts: for every object $M \in \mathcal{C}\mathcal{A}$, there is an isomorphisms in $\mathcal{C}(\mathcal{B})$ $\mathcal{F}(M[1]) \cong \mathcal{F}(M)[1]$;*

2. *$\mathcal{F}$ sends graded split exact sequences to graded split exact sequences.*

Proof. If M is a dg $\mathcal{A}$-module, there exist two obvious closed morphisms

$$i\colon M \to M[1]$$

of degree 1 and

$$j\colon M[1] \to M$$

of degree -1 such that $ij = \mathrm{id}_{M[1]}$ and $ji = \mathrm{id}_M$. The existence of these morphisms completely characterizes $M[1]$: the fact that i and j are inverses identifies $M[1]$ with M as a graded module, and the fact that they are closed characterizes the differential on $M[1]$ as the same of M (up to a sign). Since $\mathcal{F}$ is a dg-functor, $\mathcal{F}(i)$ and $\mathcal{F}(j)$ are still mutually inverse degree 1 and -1 closed morphisms, so they identify $\mathcal{F}(M[1])$ and $\mathcal{F}(M)[1]$.

To prove the second claim, we use the fact that a graded split sequence is a split exact sequence in the abelian category $\mathcal{M}od\text{-}\mathcal{A}_{gr}$, and that an additive functor between abelian categories preserves split exact sequences. □

The above lemma has an immediate corollary:

Proposition 3.51. *Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ a dg-functor between pretriangulated dg-categories. Then*

$$H^0\mathcal{F}\colon H^0\mathcal{A} \to H^0\mathcal{B}$$

is an exact functor between triangulated categories.

Although this suffices most of the times, it is still useful to know that dg-functors commute strictly (i.e. not up to homotopy) with cones.

Proposition 3.52. *Any dg-functor $\mathcal{F}\colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$ preserves cones: we have that*

$$C(\mathcal{F}(f)) \cong \mathcal{F}(C(f))$$

for any dg-natural transformation $f\colon M \to N$.

Proof. This is [BLL04, Lemma 3.8]. Since

$$\mathcal{F}_{gr}\colon \mathcal{M}od\text{-}\mathcal{A}_{gr} \to \mathcal{M}od\text{-}\mathcal{B}_{gr}$$

is an additive functor (in the sense that preserves the additive structure of the hom-spaces) it also preserves biproducts ([Mac71, VIII.2]). So, as a graded module,

$$\mathcal{F}(C(f)) = \mathcal{F}(N \oplus M[1]) \cong \mathcal{F}(N) \oplus \mathcal{F}(M[1]).$$

To prove that the differentials coincide, one considers the four natural morphisms

$$M[1] \underset{j}{\overset{p}{\leftrightarrows}} C(f) \underset{\pi}{\overset{i}{\leftrightarrows}} N$$

given by the inclusions and projections. We already know that $dp = di = 0$. On the other hand,

$$dj = i \circ f \text{ and } d\pi = -f \circ p.$$

Since $\mathcal{F}$ preserves these relations and the differential on $C(f)$ is completely characterized by them (for how one would prove this, see the proof of Lemma 3.46), the claim follows. □

Remark. The same reasoning shows the (apparently obvious) fact that if $\mathcal{A}$ is any dg-category, the category $\mathcal{M}od\text{-}\mathcal{A}$ is strongly pretriangulated: the cone (in $\mathcal{M}od\text{-}\mathcal{A}$) of a closed degree 0 morphism is isomorphic to the cone of the same morphism in $\mathcal{M}od\text{-}\mathcal{M}od\text{-}\mathcal{A}$.

Corollary 3.53. *If X is an $\mathcal{A}$-$\mathcal{B}$ bimodule, then $-\otimes_{\mathcal{A}} X$ and $\mathcal{H}om(X,-)$ induce an adjoint couple of exact functors between the triangulated categories $\mathcal{H}\mathcal{A}$ and $\mathcal{H}\mathcal{B}$.*

Proof. $-\otimes_{\mathcal{A}} X$ and $\mathcal{H}om(X,-)$ are dg-functors between the dg-categories $\mathcal{M}od\text{-}\mathcal{A}$ and $\mathcal{M}od\text{-}\mathcal{B}$, so they induce functors between the homotopy categories. By Proposition 3.51 those are exact, and by Corollary 3.38 those are adjoint (take the homology of both sides of the isomorphism). □

At this point, we would like to define the pretriangulated hull $\mathcal{A}^{pre\text{-}tr}$ of a dg-category $\mathcal{A}$ as the smallest pretriangulated subcategory of $\mathcal{C}(\mathcal{A})$ containing $\mathcal{A}$.

As it is, this definition is very intuitive but slightly problematic, since the intersection of two pretriangulated subcategories (not closed under isomorphisms) might fail to be pretriangulated. An easy fix[7] is to define explicitly the objects of $\mathcal{A}^{pre\text{-}tr}$ in the following way: to begin with, set $\mathcal{A}_0 = \mathcal{A}$. Then, define inductively the dg-category $\mathcal{A}_{n+1}$ by adding to $\mathcal{A}_n$ all the shifts and strict cones (as always, in $\mathcal{C}(\mathcal{A})$) of objects and morphisms in $\mathcal{A}_n$.

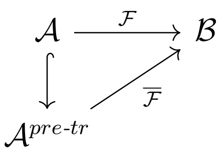
[7]but not the only one; for example we could have only considered subcategories closed under isomorphism, and defined $\mathcal{A}^{pre\text{-}tr}$ as the intersection of those, finding a slightly bigger but still dg-equivalent category.

Definition 3.54. The pretriangulated hull $\mathcal{A}^{pre\text{-}tr}$ of a dg-category $\mathcal{A}$ is defined as the full subcategory of $\mathcal{C}(\mathcal{A})$

$$\mathcal{A}^{pre\text{-}tr} = \bigcup_n \mathcal{A}_n.$$

In the following, we will use the word cone to denote either the strict cone of a morphism or its cone in the homotopy category. It is clear by the definition that $\mathcal{A}^{pre\text{-}tr}$ is a strongly pretriangulated dg-category. Similarly, $\mathcal{A}$ is strongly pretriangulated if and only if $\mathcal{A} \hookrightarrow \mathcal{A}^{pre\text{-}tr}$ is an equivalence, and is pretriangulated if and only is $H^0\mathcal{A} \hookrightarrow H^0\mathcal{A}^{pre\text{-}tr}$ is an equivalence of categories. This definition of $\mathcal{A}^{pre\text{-}tr}$ may seem unsatisfactory, since it is not *a priori* obvious that there can not exist a "smaller" pretriangulated category containing $\mathcal{A}$, living outside of the category $\mathcal{M}od\text{-}\mathcal{A}$. This is not the case; $\mathcal{A}^{pre\text{-}tr}$ can be shown to satisfy a universal property.

Proposition 3.55. *Let $\mathcal{B}$ be a strongly pretriangulated dg-category and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. Then, there exists dg-functor $\overline{\mathcal{F}}\colon \mathcal{A}^{pre\text{-}tr} \to \mathcal{B}$, unique up to isomorphism, extending $\mathcal{F}$, i.e. such that the following diagram commutes.*



Proof. The dg-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ induces the induction dg-functor

$$\mathrm{Ind}_{\mathcal{F}}\colon \mathcal{M}od\text{-}\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B},$$

extending $\mathcal{F}$. Denoting with $\overline{\mathcal{B}}$ the essential image of the dg-Yoneda embedding $\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{B}$, we prove by induction that $\mathrm{Ind}_{\mathcal{F}}(\mathcal{A}_n) \subseteq \overline{\mathcal{B}}$. To begin with, since on $\mathcal{A}_0 = \mathcal{A}$ the dg-functor $\mathrm{Ind}_{\mathcal{F}}$ coincides with $\mathcal{F}$ (up to isomorphism), we have that $\mathrm{Ind}_{\mathcal{F}}(\mathcal{A}_0) \subseteq \overline{\mathcal{B}}$. The inductive step follows from Lemma

3.52 and the fact that $\mathcal{B}$ is strongly pretriangulated; so $\operatorname{Ind}_{\mathcal{F}}(\mathcal{A}^{pre\text{-}tr}) \subseteq \overline{\mathcal{B}}$. At this point, $\overline{\mathcal{F}}$ is constructed by composing $\operatorname{Ind}_{\mathcal{F}}$ (restricted to $\mathcal{A}^{pre\text{-}tr}$) with a quasi-inverse to the inclusion $\mathcal{B} \to \overline{\mathcal{B}}$. The uniqueness comes again from Lemma 3.52. □

There exists a similar universal property where $\mathcal{B}$ is (not strongly) pretriangulated, but it requires some concepts that have not been introduced yet.

3.4.1 Enhancements of triangulated categories

We can now introduce the main topic of this thesis. We have seen that to any dg-category we can associate a canonical triangulated category; the following definition is then then very natural.

Definition 3.56. Let $\mathcal{T}$ be a (k-linear) triangulated category. A dg-enhancement of $\mathcal{T}$ is a couple $(\mathcal{A}, \epsilon)$ where $\mathcal{A}$ is a pretriangulated dg-category and

$$\epsilon \colon \mathcal{T} \xrightarrow{\sim} H^0\mathcal{A}$$

is an exact equivalence.

Often, we will refer to only $\mathcal{A}$ as the enhancement, the equivalence being implicit. As it is, this notion is not very useful. To begin with, not all triangulated categories admit an enhancement, but this will keep being true even after we refine our notions. Secondly, and more importantly, even when an enhancement exists there is no hope for it to be unique up to equivalence. The reason for this is that it is possible (and common) for a dg-functor to induce an isomorphism at the level of the homotopy category, but to fail to be an equivalence. For this reason, the correct notion to consider is not that of equivalence but of quasi-equivalence, which is roughly defined as a dg-functor inducing an equivalence at the level of the homotopy category. In order to make sense of matters of uniqueness, we have to study the homotopy theory of dg-categories. This will be done in section 3.6.

Existence of an enhancement

As already said, not all triangulated categories admit dg-enhancements. The prototypical example of a triangulated category without a dg-enhancement is the stable homotopy category, i.e. the homotopy category of spectra. This admits several equivalent definitions: we refer to [CS17, Example 3.5] for a practical definition in our setting as well as a proof that it does not admit a dg-enhancement, and to [Lur17, Section 1.4.3] for a more modern approach. The non-existence of a dg-enhancement in this case is not however particularly surprising: the stable homotopy category comes already with an enhancement, either in the form of a stable model category in the sense of [Hov07] or a stable ∞-category in the sense of [Lur17]; these are fundamentally topological objects, without any obvious linear structure.

Definition 3.57. A triangulated category is said to be algebraic if it admits a dg-enhancement.

Similarly, one says that a triangulated category is topological if it admits an enhancement of the same type of the stable homotopy category. It can be proved (see for example [Lur17, Section 1.3.1]) that all topological triangulated categories are in particular algebraic, while we have seen that the converse is not true. It should be noted that Muro, Strickland and Schwede in [MSS07] have found examples of triangulated categories that are not topological, i.e. do not admit any type of enhancements. Those however appear to be very pathological.[8]

Since this thesis focuses on dg-enhancements, we will naturally be mainly interested in algebraic categories; we will see in the following that those comprise virtually all the triangulated categories that could reasonably be defined "algebraic".

[8] Not in the sense that they are particularly hard to define, but that are very far from the platonic ideal of what a triangulated category "should" look like.

Example 3.10. The category $\mathcal{K}(\mathbf{A})$ is algebraic for any abelian category $\mathbf{A}$; a natural enhancement is given by the dg-category $\mathbf{C}_{\mathrm{dg}}(\mathbf{A})$. Similarly, the categories $\mathcal{K}^+(\mathbf{A})$, $\mathcal{K}^-(\mathbf{A})$ and $\mathcal{K}^b(\mathbf{A})$ are always algebraic. Moreover, any triangulated subcategory of an algebraic category is clearly again algebraic. We will also see in later sections that there exists a general procedure to give enhancements to Verdier quotients of algebraic triangulated categories, therefore any derived category is also algebraic.

Remark. Often, for example in geometric settings, it is useful to consider enhancements different to the canonical ones (see for example [LS16] or [CS17, Section 1.2]). In those cases, it is important to know that those enhancements are in some way equivalent; we will study similar questions in Chapter 4.

Recall that, given an additive category A, $\mathcal{K}(A)$ denotes the category of complexes of objects of A up to homotopy, and that it is in a natural way a triangulated category. We have the following characterization.

Proposition 3.58. *Let $\mathcal{T}$ be a triangulated category. The following are equivalent:*

1. *$\mathcal{T}$ is algebraic;*

2. *There exists an additive category A and a fully faithful exact functor $\mathcal{T} \to \mathcal{K}(A)$.*

Proof. We refer to [CS17, Proposition 3.1] for the proof. □

It is a very natural question to ask whether all triangulated categories that are linear over a field are algebraic. All of those appearing "in nature" do; however, examples of (very complicated) triangulated categories linear over a field that do not have a dg-enhancement have been found by Rizzardo and Van den Bergh in [RB20].

3.5 The derived category of a dg-category

In this section, we define and investigate the structure of the derived category of a dg-category.

Definition 3.59. A dg-natural transformation $\varphi\colon M \to N$ between two dg $\mathcal{A}$-modules is called a quasi-isomorphism if for every object $A \in \mathcal{A}$ the induced chain map

$$\varphi_A\colon M(A) \to N(A)$$

is a quasi-isomorphism.

Remark. As in the case of chain complexes, the isomorphisms in $\mathcal{H}\mathcal{A}$ are in particular quasi-isomorphisms.

Definition 3.60. A dg $\mathcal{A}$-module M is said to be acyclic if for every $A \in \mathcal{A}$, $M(A)$ is an acyclic complex.

Proposition 3.61. *A dg-natural transformation $\varphi\colon M \to N$ is a quasi-isomorphism if and only if its cone $C(\varphi)$ is an acyclic dg-module.*

Proof. The condition of being acyclic is a "object-wise" condition, so this follows from the analogous result about chain complexes in an abelian category. □

We denote with $\mathrm{Ac}(\mathcal{A})$ the full subcategory of $\mathcal{H}\mathcal{A}$ spanned by the acyclic dg $\mathcal{A}$-modules.

Remark. $\mathrm{Ac}(\mathcal{A})$ is a localizing subcategory of $\mathcal{H}\mathcal{A}$. To begin with, it is closed under isomorphism, since an isomorphism in $\mathcal{H}\mathcal{A}$ induces isomorphisms in homology; it closed under taking cones by Proposition 3.61, since all morphisms between acyclic modules are quasi-isomorphisms. Finally, it is closed under coproducts, since taking homology commutes with coproducts.

Definition 3.62. Let $\mathcal{A}$ be a dg-category. Its derived category $D(\mathcal{A})$ is defined as the Verdier quotient

$$D(\mathcal{A}) = {}^{\mathcal{HA}}\!\big/_{\mathrm{Ac}(\mathcal{A})}.$$

Equivalently, $D(\mathcal{A})$ is defined the localization of $\mathcal{HA}$ with respect to quasi-isomorphisms.

Remark. If $\mathcal{A}$ is small, the derived category $D(\mathcal{A})$ is compactly generated by the set $\{h_A\}_{A\in\mathcal{A}}$. Indeed, let M be a dg $\mathcal{A}$-module. The dg-modules h_A are compact by the Yoneda lemma. Suppose that

$$\mathrm{Hom}_{\mathcal{HA}}(h_A, M[n]) = 0$$

for any $A \in \mathcal{A}$ and $n \in \mathbb{Z}$. This, by the dg-Yoneda lemma, implies that

$$H^n M(A) = 0$$

for any $A \in \mathcal{A}$ and $n \in \mathbb{Z}$. This is equivalent to saying that M is acyclic, therefore M is isomorphic to 0 in the derived category. We will also prove in section 3.5.1 that it is possible to explicitly express any module up to quasi-isomorphism as the cone of a morphism between free modules. Of course, even if $\mathcal{A}$ is not small, the objects of the form h_A are compact and generate $D(\mathcal{A})$.

Proposition 3.63. *Let*

$$0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$$

be an exact sequence in $\mathcal{CA}$. It can then be completed to an exact triangle

$$M \xrightarrow{f} N \to L \to N[1]$$

in $D(\mathcal{A})$.

Proof. We construct a quasi-isomorphism $C(f) \to N$. Consider the diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \longrightarrow & C(f) \\ \| & & \| & & \downarrow{\scriptstyle\varphi} \\ M & \xrightarrow{f} & N & \xrightarrow{g} & L \end{array}$$

Since $C(f) = M[1] \oplus N$ as a graded module, we define $\varphi(x^{n+1}, y^n) = g(y_n)$. By the fact that $gf = 0$, φ commutes with the differentials. Since g is surjective, φ is surjective as well. Therefore we have an exact sequence in $\mathcal{CA}$

$$0 \to \operatorname{Ker}\varphi \to C(f) \xrightarrow{\varphi} L \to 0;$$

To prove that φ is a quasi-isomorphism is then equivalent to showing that $\operatorname{Ker}\varphi$ is ayciclic. By definition of φ, we have that $\operatorname{Ker}\varphi$ coincides with the cone of the map

$$M \xrightarrow{f} \operatorname{Im} f.$$

Since this is an isomorphism, $\operatorname{Ker}\varphi$ is acyclic. □

Remark. As in the case of $D(\mathbf{A})$ for an abelian category $\mathbf{A}$, if we have a sequence of morphisms in $\mathcal{C}(\mathcal{A})$

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \to \cdots$$

we can define the object $\underrightarrow{\operatorname{hocolim}}_i X_i$ in $D(\mathcal{A})$, and there is an isomorphism in $D(\mathcal{A})$

$$\underset{i}{\underrightarrow{\operatorname{hocolim}}}\, X_i \xrightarrow{\sim} \underrightarrow{\operatorname{colim}}\, X_i.$$

Definition 3.64. A dg $\mathcal{A}$-module P is said to be h-projective if

$$\operatorname{Hom}_{\mathcal{HA}}(P, N) = 0$$

for any acyclic dg $\mathcal{A}$-module N.

We will denote with h-proj$(\mathcal{A})$ the full subcategory of $\mathcal{M}od\text{-}\mathcal{A}$ spanned by the h-projective modules. In the language of Chapter 1, the subcategory h-proj$(\mathcal{A}) \subseteq \mathcal{HA}$ coincides with the orthogonal subcategory ${}^{\perp}\mathrm{Ac} \subseteq \mathcal{HA}$. Note that by Proposition 1.31, if P is h-projective the natural map

$$\operatorname{Hom}_{\mathcal{HA}}(P, M) \to \operatorname{Hom}_{D(\mathcal{A})}(P, M)$$

is an isomorphism for any dg $\mathcal{A}$-module M. At this point, we want to prove the existence of h-projective resolutions for any dg $\mathcal{A}$-module, i.e. for any dg $\mathcal{A}$-module M, a h-projective module P together with a quasi-isomorphism $P \xrightarrow{\sim} M$. Since the cone of a quasi-isomorphism is acyclic, we will then be able to apply Proposition 1.32 to find a fully faithful functor

$$\mathbf{p} \colon D(\mathcal{A}) \to \mathcal{HA}$$

that is a right adjoint to the quotient $\mathcal{HA} \to D(\mathcal{A})$. To do this we will restrict to a subclass of h-proj$(\mathcal{A})$ that is particularly amenable to computations, the semi-free dg $\mathcal{A}$-modules.

Definition 3.65. A dg $\mathcal{A}$-module M is said to be semi-free if it admits a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq M$$

such that the following conditions are satisfied:

1. $M = \bigcup_i F_i = \varinjlim_i F_i$;
2. The quotients F_{i+1}/F_i are free dg $\mathcal{A}$-modules.

Remark. The filtration in the definition is indexed by an arbitrary ordinal, not necessarily by the integers.

Lemma 3.66. *Any exact sequence of the form*

$$0 \to M \xrightarrow{i} N \xrightarrow{p} L \to 0$$

where L is a free dg $\mathcal{A}$-module is graded split.

Proof. We find a splitting to p. We know that

$$L \cong \bigoplus_i h_{A_i}[n_i],$$

so

$$\mathcal{N}at_{\mathrm{dg}}(L, N) \cong \prod_i \mathcal{N}at_{\mathrm{dg}}(h_{A_i}[n_i], N) \cong \prod_i \mathcal{N}at_{\mathrm{dg}}(h_{A_i}, M)[n_i],$$

and we can reduce to the case where $L \cong h_A$. Since p is surjective, we can find an element $x \in M(A)^0$ such that $p_A(x) = \mathrm{id}_A$. Now the dg-Yoneda lemma gives a graded natural transformation $\varphi\colon h_A \to N$ of degree 0 such that $\varphi_A(\mathrm{id}_A) = x$. Notice that, since x may not be closed, φ may fail to preserve the differential. Another application of the dg-Yoneda lemma shows that $p \circ \varphi = \mathrm{id}_{h_A}$, since $p_A \circ \varphi_A(\mathrm{id}_A) = \mathrm{id}_A$, so φ is a splitting of p. □

The above result implies that is is always possible to suppose that the inclusions $F_i \hookrightarrow F_{i+1}$ in the definition of semi-free dg $\mathcal{A}$-module admit a splitting (that may not preserve the differential).

Lemma 3.67. *Let M be a semi-free dg $\mathcal{A}$-module. There is a graded split short exact sequence in $\mathcal{C}(\mathcal{A})$*

$$0 \to \bigoplus_i F_i \xrightarrow{\alpha} \bigoplus_i F_i \xrightarrow{\pi} M \to 0 \tag{3.19}$$

with F_i as in the definition of semi-free module.

Proof. We have to define α and π; denote with ι the inclusion $F_i \hookrightarrow F_{i+1}$. For $x_i \in F_i$, define

$$\alpha_i(x_i) = x_i - \iota(x_i) \in F_i \oplus F_{i+1},$$

and α as the sum of the α_i; π is defined as the sum of the inclusions. It's clear that $\pi \circ \alpha = 0$; on the other hand, given a formal sum $x = \sum_i x_i$ with $x_i \in F_i$ such that $\pi(x) = 0$, setting $x_{-1} = 0$ and defining

$$y = \sum_i x_i + \iota(x_{i-1}) + \iota(x_{i-2}) + \ldots + \iota(x_0)$$

we get $x = \alpha(y)$. In order to prove that the sequence is graded split we find a left inverse to α. Denote with $s_i \colon F_i \to F_{i-1}$ a splitting to the inclusion. For a given $x_i \in F_i$, define

$$\beta_i(x_i) = \sum_{j \leq i} s_j \circ s_{j+1} \circ \ldots \circ s_i(x_i)$$

and β as the sum of the β_i. β is seen to be a left inverse to α, so the sequence is graded split. □

Proposition 3.68. *A semi-free dg $\mathcal{A}$-module is h-projective.*

Proof. By the dg-Yoneda lemma, a free module is h-projective: indeed, for any acyclic module N,

$$\mathcal{HA}(h_A[n], N) \cong H^{-n}(N) = 0$$

for any object A. By the universal property of a coproduct, this implies that free modules are h-projective. Now let M be semi-free, and let

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq M$$

be a filtration as per Definition 3.65. By induction, each F_i is h-projective: F_0 is free and for every i the short exact sequence

$$0 \to F_i \to F_{i+1} \to F_{i+1}/F_i \to 0$$

is graded split, so by Proposition 3.46 defines an exact triangle in $\mathcal{HA}$. Then, since $\mathrm{Hom}_{\mathcal{HA}}(-, N)$ is cohomological, if F_i is h-projective then F_{i+1} is as well. Finally, the same reasoning of the inductive step applied to the triangle induced by the sequence (3.19) gives that M is h-projective. □

Corollary 3.69. *The derived Yoneda embedding induces, by composition with the quotient $\mathcal{HA} \to D(\mathcal{A})$, a fully faithful functor*

$$H^0\mathcal{A} \to D(\mathcal{A}).$$

We will also call this functor the derived Yoneda embedding.

Proof. Since representable modules are h-projective, we have that

$$H^0\mathcal{A}(A, B) \to \mathcal{HA}(h_A, h_B) \to D(\mathcal{A})(h_A, h_B)$$

is an isomorphism for all $A, B \in \mathcal{A}$. $\square$

Lemma 3.70. *Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a dg-functor. Then,*

$$\mathrm{Ind}_{\mathcal{F}}\,\mathcal{SF}(\mathcal{A}) \subseteq \mathcal{SF}(\mathcal{B}).$$

Proof. We have already seen that $\mathrm{Ind}_{\mathcal{F}}$ carries representable modules to representable modules, and preserves shifts. Since $\mathrm{Ind}_{\mathcal{F}}$ is a left adjoint, it preserves coproducts and therefore preserves free modules. Then, let P be a semi-free dg $\mathcal{A}$-module and

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq P$$

the relative filtration. We prove by induction that the filtration

$$0 = \mathrm{Ind}_{\mathcal{F}}(F_{-1}) \subseteq \mathrm{Ind}_{\mathcal{F}}(F_0) \subseteq \mathrm{Ind}_{\mathcal{F}}(F_1) \subseteq \cdots \subseteq \mathrm{Ind}_{\mathcal{F}}(P)$$

presents $\mathrm{Ind}_{\mathcal{F}}(P)$ as a semi-free dg $\mathcal{A}$-module. By the discussion above, $\mathrm{Ind}_{\mathcal{F}}(F_0)$ is free. By definition of semi-free, there is a short exact sequence in $\mathcal{CA}$

$$0 \to F_i \to F_{i+1} \to F_{i+1}/F_i \to 0$$

that by Lemma 3.66 is graded split. Since any additive functor sends split exact sequences to split exact sequence,

$$0 \to \mathrm{Ind}_{\mathcal{F}}(F_i) \to \mathrm{Ind}_{\mathcal{F}}(F_{i+1}) \to \mathrm{Ind}_{\mathcal{F}}(F_{i+1}/F_i) \to 0$$

is split exact, so in particular exact. Since $\mathrm{Ind}_{\mathcal{F}}(F_{i+1}/F_i)$ is free, the induction step is proved. $\square$

Similarly, one easily proves that $\mathrm{Ind}_{\mathcal{F}}$ sends h-projective modules to h-projective modules.

Remark. If $f\colon M \to N$ is a (closed, degree 0) morphism between semi-free dg $\mathcal{A}$-modules, then the cone $C(f)$ is again semi-free. To see this, suppose that M admits a filtration

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq M$$

and N admits a filtration

$$E_0 \subseteq E_1 \subseteq \cdots \subseteq N.$$

Then the filtration

$$E_0 \subseteq E_1 \subseteq \cdots \subseteq N \subseteq N \oplus E_1[1] \subseteq \cdots \subseteq N \oplus M[1] = C(f)$$

shows that $C(f)$ is semi-free as well. Note that in this case it was important for the filtration to be indexed by an arbitrary ordinal. More easily, the fact that the cone of a morphism between h-projective modules is h-projective follows immediately from the fact that the functor $\mathrm{Hom}_{\mathcal{HA}}(-, N)$ is cohomological.

Remark. Reasoning as in the case of $\mathcal{M}od\text{-}\mathcal{A}$, the above remark gives that the dg-categories $\mathcal{SF}(\mathcal{A})$ and h-proj$(\mathcal{A})$ are strongly pretriangulated.

3.5.1 Existence of semi-free resolutions

The goal of this section is to prove the following fact:

Proposition 3.71. *Let M be a dg $\mathcal{A}$-module. Then, there exists a quasi-isomorphism $P \xrightarrow{\eta} M$, where P is a semi-free dg $\mathcal{A}$-module. Furthermore, η can be chosen to be surjective.*

In particular, we have a well defined fully faithful right adjoint

$$\mathbf{p}\colon \mathcal{D}(\mathcal{A}) \to \mathcal{HA}$$

to the quotient $\mathcal{HA} \to D(\mathcal{A})$, defined by choosing, once and for all, a semi-free "resolution" for each element of $D(\mathcal{A})$. Moreover, the composition

$$H^0\mathcal{SF}(\mathcal{A}) \hookrightarrow \mathcal{HA} \to D(\mathcal{A})$$

is an exact equivalence. Therefore, $\mathcal{SF}(\mathcal{A})$ and, equivalently, h-proj$(\mathcal{A})$ are dg-enhancements of the triangulated category $D(\mathcal{A})$.

For the proof, we begin with the promised generalization of Corollary 3.32.

Lemma 3.72. *Let M be a dg $\mathcal{A}$-module. Then there exists a semi-free dg $\mathcal{A}$-module P and a dg-natural transformation $P \to M$ such than for any $n \in \mathbb{Z}$ and for any $A \in \mathcal{A}$, the induced maps*

$$Z^n P(A) \to Z^n M(A)$$

and

$$P(A)^n \to M(A)^n$$

are surjective.

Proof. Recalling the proof of Corollary 3.32, it is enough to find for every $A \in \mathcal{A}$ and for every $x \in M(A)$ a semi-free dg $\mathcal{A}$-module C_A and a dg-natural transformation $\eta \colon C_A \to M$ such that $x \in \mathrm{Im}(\eta_A)$. Suppose for simplicity that $x \in M(A)^{-1}$, as the general following from this by "shifting" the whole argument. Define the dg $\mathcal{A}$-module

$$C_A = C(h_A \xrightarrow{\mathrm{id}} h_A) = h_A[1] \oplus h_A,$$

with

$$d(\alpha^{n+1}, \beta^n) = (-d\alpha^{n+1}, d\beta^n + \alpha^{n+1}).$$

By the dg-Yoneda Lemma there are two degree 0 graded natural transformations

$$\varphi \colon h_A[1] \to M$$

such that $\varphi_A(\mathrm{id}_A) = x$ and

$$\psi \colon h_A \to M$$

such that $\psi_A(\mathrm{id}_A) = dx$. Again by the dg-Yoneda Lemma, since dx is a closed element, $d(\psi) = 0$ while $d(\varphi) = \psi$[9]. We define $\eta \colon C_A \to M$ as the sum of φ and ψ. We know that η is a graded natural transformation of degree 0, but it is not obvious that is commutes with the differentials. However,

$$\begin{aligned}\eta d(\alpha^{n+1}, \beta^n) =& \eta(-d\alpha^{n+1}, d\beta^n + \alpha^{n+1}) = -\varphi(d\alpha^{n+1}) + \psi(d\beta^n) + \psi(\alpha^{n+1}) = \\ =& d\varphi(\alpha^{n+1}) + d\psi(\beta^n) = d\eta(\alpha^{n+1}, \beta^n),\end{aligned}$$

where we have crucially used the fact that

$$\varphi(d\alpha^{n+1}) = (d\varphi)(\alpha^{n+1}) - d(\varphi\alpha^{n+1}) = \psi(\alpha^{n+1}) - d(\varphi\alpha^{n+1}).$$

That C_A is semi-free follows from the filtration $0 \subseteq h_A \subseteq C_A$. At this point, one concludes by taking sums of shifts of modules of the form C_A (to obtain the surjectivity) and h_A (to get the surjectivity on the cycles), as in the proof of Corollary 3.32. The (two step) filtrations on each summand induce a suitable two-step filtration on P, so P is itself semi-free. □

[9] This is a slight abuse of notation, but is makes sense in light of the isomorphism (3.3)

Lemma 3.73. *Let M be a dg $\mathcal{A}$-module. Then there exists an exact sequence in $\mathcal{C}(\mathcal{A})$*

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \to M \to 0$$

such that P_{-i} are semi-free dg $\mathcal{A}$-modules and

$$\cdots \to Z^n P_{-2}(A) \to Z^n P_{-1}(A) \to Z^n P_0(A) \to Z^n M(A) \to 0$$

is exact for every $A \in A$. These two conditions also imply that

$$\cdots \to H^n P_{-2}(A) \to H^n P_{-1}(A) \to H^n P_0(A) \to H^n M(A) \to 0$$

is exact for every $A \in \mathcal{A}$.

Proof. This is done in the same way in which one proves the existence of projective resolutions in abelian categories with enough projectives; first, Lemma 3.5.1 gives a surjection $P_0 \to M$ that is also surjective when restricted to the cycles. Denoting with M_0 the kernel of the surjection, we have an exact sequence in $\mathcal{C}(\mathcal{A})$

$$0 \to M_0 \to P_0 \to M \to 0$$

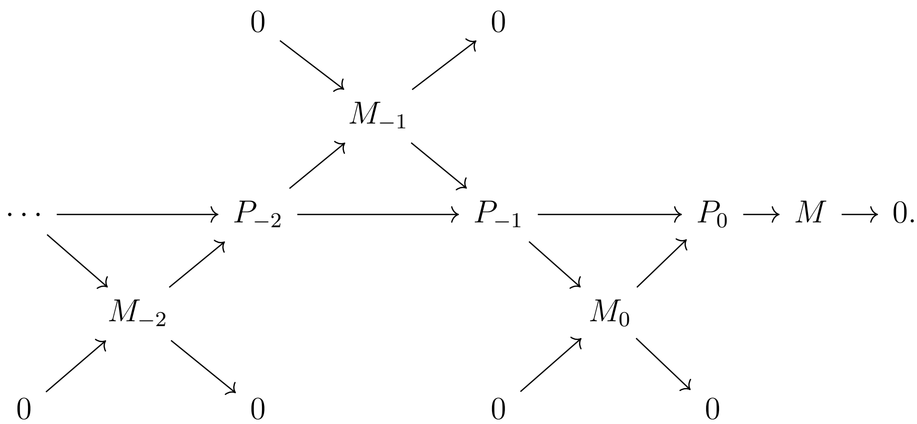
that is also exact when restricted to the cycles. Applying again Lemma 3.5.1 to M_0 and denoting with M_{-1} the kernel of this surjection, we get another exact sequence (also exact at the cycles)

$$0 \to M_{-1} \to P_{-1} \to M_0 \to 0.$$

This way, we can inductively find exact sequences

$$0 \to M_{-i} \to P_{-i} \to M_{-i+1} \to 0$$

that are also exact at the cycles, and finally find the long exact sequence by gluing those according to the diagram



To prove that

$$\cdots \to H^n P_{-2}(A) \to H^n P_{-1}(A) \to H_n^P(A) \to H^n M(A) \to 0$$

is exact, we prove that the exactness of the sequences

$$0 \to M_{-i}(A)^n \to P_{-i}(A)^n \to M_{-i+1}(A)^n \to 0$$

and

$$0 \to Z^n M_{-i}(A) \to Z^n P_{-i}(A) \to Z^n M_{-i+1} \to 0$$

imply that

$$0 \to H^n M_{-i}(A) \to H^n P_{-i}(A) \to H^n M_{-i+1} \to 0$$

is exact. In order to do this, recall that by definition of cycles and boundaries, for any chain complex A and any n we have a short exact sequence

$$0 \to Z^n A \to A^n \to B^{n+1} A \to 0;$$

assembling this and the two above into the diagram below

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^n M_{-i}(A) & \longrightarrow & Z^n P_{-i}(A) & \longrightarrow & Z^n M_{-i+1}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{-i}(A)^n & \longrightarrow & P_{-i}(A)^n & \longrightarrow & M_{-i+1}(A)^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^{n+1} M_{-i}(A) & \longrightarrow & B^{n+1} P_{-i}(A) & \longrightarrow & B^{n+1} M_{-i+1}(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the snake lemma gives us that since the first two rows are exact, the third is as well. Applying the same argument using now the short exact sequence

$$0 \to B^n A \to Z^n A \to H^n A \to 0$$

gives, again by the snake lemma, that

$$0 \to H^n M_{-i}(A) \to H^n P_{-i}(A) \to H^n M_{-i+1} \to 0$$

is exact. So the short sequences that get glued into the long sequence are also exact in homology, so the long sequence is as well. □

Before concluding the proof of Proposition 3.71, we give a construction. Suppose we have a chain complex of objects in $C(\mathcal{A})$, that is a sequence of morphisms

$$\cdots P_{-1} \xrightarrow{d_h} P_0 \xrightarrow{d_h} P_1 \to \cdots \tag{3.20}$$

such that $d_h^2 = 0$[10]. Denote with $d_v \colon P_i(A)^n \to P_i(A)^{n+1}$ the differential of the complex $P_i(A)$. We can define the total dg $\mathcal{A}$-module $\operatorname{Tot} P_\bullet$ this way: given $A \in \mathcal{A}$, define

$$\operatorname{Tot}(P)(A)^n = \bigoplus_{i+j=n} P_i(A)^j$$

as a graded module, with differential defined for $x \in P_i(A)^j$ as

$$dx = d_h x + (-1)^i d_v x.$$

At this point, one proves that $\operatorname{Tot}(P_\bullet)$ is a dg $\mathcal{A}$-module similarly to how we proved that the cone a morphism is a dg $\mathcal{A}$-module. Graphically, to visualize $\operatorname{Tot}(P_\bullet)$ one considers the sequence (3.20) as a grid

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots \\
\uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} \\
\cdots & \xrightarrow{d_h} & P^1_{-1} & \xrightarrow{d_h} & P^1_0 & \xrightarrow{d_h} & P^1_{-1} & \xrightarrow{d_h} & \cdots \\
\uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} \\
\cdots & \xrightarrow{d_h} & P^0_{-1} & \xrightarrow{d_h} & P^0_0 & \xrightarrow{d_h} & P^0_1 & \xrightarrow{d_h} & \cdots \\
\uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} \\
\cdots & \xrightarrow{d_h} & P^{-1}_{-1} & \xrightarrow{d_h} & P^{-1}_0 & \xrightarrow{d_h} & P^{-1}_1 & \xrightarrow{d_h} & \cdots \\
\uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} & & \uparrow{\scriptstyle d_v} \\
\cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \cdots
\end{array}$$

so that each $\operatorname{Tot}(P_\bullet)^n$ is defined as the direct sum of a diagonal, and the differential sends an homogeneous object to the next diagonal by pushing it both upwards and rightward (up to a sign which is necessary to get that $d^2 = 0$).

Remark. When dealing with complexes of this kind, we will keep denoting with $Z^n P_i$, $B^n P_i$ and $H^n P_i$ the cycles, boundaries and homologies of the i-th module P_i (according to the vertical differentials).

Proof of Proposition 3.71. We construct P explicitly. Recall the exact sequence of Lemma 3.73

$$\cdots \to P_{-2} \to P_{-1} \to P_0 \to M \to 0.$$

[10] The subscript h in d_h stands for horizontal, for reasons that will soon be clear. Also, several subscript have and will be suppressed for the sake of reading clarity.

Denote now with d_h the maps $P_{-i} \to P_{-i+1}$, with d_v the differential of P_{-i} and consider the complex in $\mathcal{C}(\mathcal{A})$

$$\cdots \xrightarrow{d_h} P_{-2} \xrightarrow{d_h} P_{-1} \xrightarrow{d_h} P_0 \to 0.$$

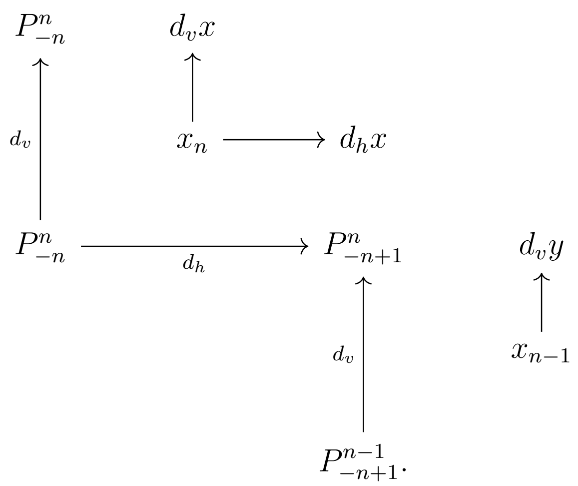
Define $P = \operatorname{Tot} P_\bullet$. The map $P_0 \to M$ induces a map $P \to M$: write an element $x \in P^n$ as a sum

$$x = \sum x_i$$

with $x_i \in P^{i+n}_{-i}$. Sending x to the image of x_0 via the map $P_0 \to M$ gives a well-defined dg-natural transformation $P \to M$. We will have finished if we prove that $P \to M$ is a quasi-isomorphism and that P is semi-free. Let's begin with the first claim: we prove that $H^0 P \to H^0 M$ is an isomorphism[11], the case $n \neq 0$ being identical (but with slightly more confusing indices). Just for the proof of this theorem, we will call d_h (resp. d_v or d)-cycles the elements in the kernel of d_h (resp. of d_v or of d), and similarly for boundaries. Let $x \in P^0$ be a d-cycle. As before, write x as a sum of homogeneous elements

$$x = \sum x_i$$

with $x_i \in P^i_{-i}$. We now want to show that we can change x by a d-boundary in a way that the only non-zero element is the sum becomes $x_0 \in P^0_0$. Let n be the largest integer such that $x_i \neq 0$. Since $dx = 0$, we have that $d_v(x_n) = 0$ (since it would be the only component of dx in P^{n+1}_{-n}) and $d_h(x_n)$ is, up to a sign, equal to $d_v(x_{n-1})$ (since they are the only elements of dx in P^n_{-n+1}). This is exemplified graphically by the diagram



The first condition tells us that x_n represents a class in $H^n P_{-n}$, and the second that the image of this class in $H^n P_{-n+1}$ via d_h is zero, since it is sent

[11] Recall that by this we mean that $H^0 P(A) \to M(A)$ is an isomorphism for all $A \in \mathcal{A}$. In this type of proofs this abuse of notation is particularly harmless, since the condition that we want to prove is "object-wise" and there is no hidden naturality condition.

to a d_v-boundary. Therefore, by exactness, x_n is, up to a d_v-boundary, equal to the image via d_h of a d_v-cycle. This means that there exist $y \in P_{-n}^{n-1}$ and $z \in Z^n P_{-n-1}$ such that

$$x_n = d_h z + d_v y.$$

Since z is a d_v-cycle, $dz = d_h z$. Then, by changing x by dz, we get that $x_n = d_v y$. Similarly, since dy coincides up to a sign and up to an element of P_{-n+1}^{-n+1} with $d_v y$, we can change again x by dy (possibly changed by a sign) and find that $x_n = 0$. By induction, we find that every d-cycle is, up to a boundary, an element $x_0 \in P_0^0$.

That $H^0 P \to H^0 M$ is surjective follows directly from the fact that $Z^0 P_0 \to Z^0 M$ is surjective. Take now $x \in H^0 P$ such that x maps to 0 in $H^0 M$. We have just proved that x admits a representative $x_0 \in P_0^0$. Since x_0 is a d-cycle, it is also a d_v-cycle, so $x_0 \in Z^0 P_0$. At this point, since

$$H^0 P_{-1} \xrightarrow{d_h} H^0 P_0 \to H^0 M$$

is exact and x_0 maps to 0 via the second map, x_0 is, up to a d_v-boundary, equal to $d_h y$ for some $y \in Z^0 P_{-1}$. Now, since for elements of P_0 being d_v-boundaries is equivalent to being d-boundaries, x_0 is, up to a d-boundary, equal to $d_h y$. However, since y is a d_v-cycle, $d_v y = dy$, so $d_v y$ is itself a d-boundary and we are done.

We have only left to prove that P is semi-free. Recall that each P_{-i} has a two-step filtration $0 \subseteq Q_{-i} \subseteq P_{-i}$. This induces a filtration of P

$$\begin{aligned}
&F_0 = 0 \\
&F_1 = P_0 \\
&F_2 = P_0 \oplus Q_0 \\
&F_3 = P_0 \oplus P_{-1} \\
&F_4 = P_0 \oplus P_{-1} \oplus Q_{-1} \\
&\dots
\end{aligned}$$

than can be used to show that P is semi-free. Beware that the differential on F_i is that induced by P, and not by the direct sum. This is unavoidable since we want F_i to be a subcomplex of P. $\square$

3.5.2 h-injective resolutions

It is useful to know that there also exists a left adjoint to the quotient, given by h-injective resolutions.

Definition 3.74. A dg A-module I is said to be h-injective if

$$\mathrm{Hom}_{\mathcal{HA}}(N, I) = 0$$

for any acyclic dg $\mathcal{A}$-module N.

The subcategory of h-injective modules is denotes with h-inj($\mathcal{A}$). By definition, we have

$$\text{h-inj}(\mathcal{A}) = \mathrm{Ac}^{\perp} \subseteq \mathcal{HA}.$$

As in the case of h-projective modules, we have

Theorem 3.75. *Let M be a dg $\mathcal{A}$-module. Then, there exists a quasi-isomorphism $M \xrightarrow{\sim} I$, where I is an h-injective dg $\mathcal{A}$-module.*

In particular, we have a well defined fully faithful left adjoint

$$\mathbf{i} \colon \mathcal{D}(\mathcal{A}) \to \mathcal{HA}$$

to the quotient $\mathcal{HA} \to D(\mathcal{A})$, defined by choosing, once and for all, an h-injective resolution for each element of $D(\mathcal{A})$.

The proof of this fact is conceptually dual to that of Proposition 3.71, but it is made slightly more difficult by the fact that representable and free modules are always h-projective, but not always h-injective. Therefore one has to define a different "building block" for the resolution, and then proceed dually to Proposition 3.71. We omit the construction, referring to [Sta, Section 22.21] for all the details.

3.5.3 Derived Hom and tensor

Recall that an $\mathcal{A}$-$\mathcal{B}$ bimodule X induces an adjoint couple

$$- \otimes_{\mathcal{A}} X : \mathcal{HA} \rightleftarrows \mathcal{HB} : \mathcal{H}om(X, -).$$

Those functors in general do not preserve quasi-isomorphisms, so do not descend in the obvious way to functors at the level of the homotopy categories. However, h-projective and h-injective resolution functors allow us to define their "best approximation", in the sense of derived functors. For simplicity of notation, set $F = - \otimes_{\mathcal{A}} X$ and $G = \mathcal{H}om(X, -)$. We can then define the derived functors

$$\mathbf{L}F : D(\mathcal{A}) \rightleftarrows D(\mathcal{B}) : \mathbf{R}G$$

by defining $\mathbf{L}F$ as the composition

$$D(\mathcal{A}) \xrightarrow{\mathbf{p}} \mathcal{HA} \xrightarrow{F} \mathcal{HB} \to D(\mathcal{B})$$

and $\mathbf{R}G$ as the composition

$$D(\mathcal{B}) \xrightarrow{\mathbf{i}} \mathcal{H}\mathcal{A} \xrightarrow{G} \mathcal{H}\mathcal{A} \to D(\mathcal{A}),$$

where in both cases the last arrow is the quotient functor.

Proposition 3.76. *The functors*

$$\mathbf{L}F : D(\mathcal{A}) \rightleftarrows D(\mathcal{B}) : \mathbf{R}G$$

form an adjoint pair.

Proof. Fix $M \in D(\mathcal{A}), N \in D(\mathcal{B})$. Then,

$$\begin{aligned} &\mathrm{Hom}_{D(\mathcal{B})}(\mathbf{L}F\, M, N) = \mathrm{Hom}_{D(\mathcal{B})}(F\,\mathbf{p}M, N) \cong \mathrm{Hom}_{\mathcal{H}\mathcal{B}}(F\,\mathbf{p}M, \mathbf{i}N) \cong \\ &\cong \mathrm{Hom}_{\mathcal{H}\mathcal{A}}(\mathbf{p}M, G\,\mathbf{i}N) \cong \mathrm{Hom}_{D(\mathcal{A})}(M, G\,\mathbf{i}N) = \mathrm{Hom}_{D(\mathcal{A})}(M, \mathbf{R}G\, N). \end{aligned}$$

□

In the following, we will sometimes denote the functor $\mathbf{L}F$ as

$$- \otimes^L X \colon D(\mathcal{A}) \to D(\mathcal{B}).$$

If $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ is a dg-functor, by setting $F = \mathrm{Ind}_{\mathcal{F}}$ and $G = \mathrm{Res}_{\mathcal{F}}$ we get an adjunction of exact functors

$$\mathbf{L}F : D(\mathcal{A}) \rightleftarrows D(\mathcal{B}) : \mathbf{R}G.$$

3.6 The homotopy theory of dg-categories

Definition 3.77. A dg-functor $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ between dg-categories is said to be a quasi-equivalence if the following two conditions are verified:

- For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the induced chain map

  $$\mathcal{A}(A, B) \to \mathcal{B}(\mathcal{F}A, \mathcal{F}B)$$

  is a quasi-isomorphism.

- The induced functor

  $$H^0\mathcal{F} \colon H^0\mathcal{A} \to H^0\mathcal{B}$$

  is an equivalence of categories.

A dg-functor satisfying only the first condition is said to be quasi-fully faithful.

The second condition "almost" implies the first one, since it implies that

$$H^0\mathcal{A}(A,B) \to H^0\mathcal{B}(\mathcal{F}A,\mathcal{F}B)$$

is an isomorphism. In order to deduce from this that

$$H^n\mathcal{A}(A,B) \to H^n\mathcal{B}(\mathcal{F}A,\mathcal{F}B)$$

is an isomorphism for an arbitrary n it is enough to suppose that either $\mathcal{A}$ or $\mathcal{B}$ admit shifts: in that case, just apply the isomorphism to $A' = A[-n]$ or $B' = B[n]$.

A dg-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ between dg-categories induces a dg-functor $\mathcal{F}\colon \mathcal{A}^{pre\text{-}tr} \to \mathcal{B}^{pre\text{-}tr}$. To see this, compose $\mathcal{F}$ with the inclusion $\mathcal{B} \hookrightarrow \mathcal{B}^{pre\text{-}tr}$ and the apply the universal property of the pretriangulated hull.

Proposition 3.78. *If $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then*

$$\mathcal{F}\colon \mathcal{A}^{pre\text{-}tr} \to \mathcal{B}^{pre\text{-}tr}$$

is a quasi-equivalence.

Proof. This is conceptually analogous to Lemma 1.19. Fix $B \in \mathcal{A}$ and consider the full subcategory $\mathcal{C} \subseteq \mathcal{A}^{pre\text{-}tr}$ of all objects A such that

$$\mathcal{A}^{pre\text{-}tr}(A,B) \to \mathcal{B}^{pre\text{-}tr}(\mathcal{F}A,\mathcal{F}B)$$

is a quasi-isomorphism. It is clear that $\mathcal{A} \subseteq \mathcal{C}$. We prove that $\mathcal{C}$ is strongly pretriangulated, which will imply that $\mathcal{C} = \mathcal{A}^{pre\text{-}tr}$. First the shifts: for $A \in \mathcal{C}$, we have

$$\begin{aligned}&\mathcal{A}^{pre\text{-}tr}(A[1],B) \cong \mathcal{A}^{pre\text{-}tr}(A,B)[-1] \xrightarrow{\sim} \mathcal{B}^{pre\text{-}tr}(\mathcal{F}A,\mathcal{F}B)[-1] \cong \\ &\cong \mathcal{B}^{pre\text{-}tr}(\mathcal{F}A[1],\mathcal{F}B),\end{aligned}$$

where the third arrow is a quasi-isomorphism; so $A[1] \in \mathcal{C}$. To prove that $\mathcal{C}$ is closed under taking cones, let $A, A' \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}\mathcal{A}}(A,A')$ be two arbitrary objects and a (closed, degree 0) morphism between them. We then have a distinguished triangle in $H^0\mathcal{A}^{pre\text{-}tr}$

$$A \to A' \to C(f) \to A[1]$$

which, writing $\mathrm{Hom}(-,-)$ for $\mathrm{Hom}_{H^0\mathcal{A}^{pre\text{-}tr}}(-,-)$, induces an exact sequence

$$\mathrm{Hom}(A'[1],B) \to \mathrm{Hom}(A[1],B) \to \mathrm{Hom}(C(f),B) \to \mathrm{Hom}(A',B) \to \mathrm{Hom}(A,B)$$

This, together with the five-lemma, implies that

$$H^0\mathcal{A}^{\textit{pre-tr}}(C(f),B) \to H^0\mathcal{B}^{\textit{pre-tr}}(\mathcal{F}C(f),\mathcal{F}B)$$

is an isomorphism. In order to deduce from this that that

$$H^n\mathcal{A}^{\textit{pre-tr}}(C(f),B) \cong H^n\mathcal{B}^{\textit{pre-tr}}(\mathcal{F}C(f),\mathcal{F}B)$$

for any n, we use that $\mathcal{C}$ is stable under shifts, so

$$\begin{aligned} H^n\mathcal{A}^{\textit{pre-tr}}(C(f),B) \cong H^0\mathcal{A}^{\textit{pre-tr}}(C(f)[n],B) \cong H^0\mathcal{A}^{\textit{pre-tr}}(C(f[n]),B) \xrightarrow{\sim} \\ \xrightarrow{\sim} H^0\mathcal{B}^{\textit{pre-tr}}(\mathcal{F}C(f)[n],\mathcal{F}B) \cong H^n\mathcal{B}^{\textit{pre-tr}}(\mathcal{F}C(f),\mathcal{F}B). \end{aligned}$$

So we have proved that for any $A \in \mathcal{A}^{\textit{pre-tr}}$ and $B \in \mathcal{A}$,

$$\mathcal{A}^{\textit{pre-tr}}(A,B) \to \mathcal{B}^{\textit{pre-tr}}(\mathcal{F}A,\mathcal{F}B)$$

is a quasi-isomorphism. To prove that this is valid for any $B \in \mathcal{B}^{\textit{pre-tr}}$, just fix any $A \in \mathcal{A}^{\textit{pre-tr}}$ and repeat the argument above considering the full subcategory of $\mathcal{B}^{\textit{pre-tr}}$ of all elements for which $\mathcal{A}^{\textit{pre-tr}}(A,B) \to \mathcal{B}^{\textit{pre-tr}}(\mathcal{F}A,\mathcal{F}B)$ is a quasi-isomorphism: we have just proved that this contains $\mathcal{A}$, and as above we prove that it is closed under shifts and cones; this proves that $\mathcal{F}$ is quasi-fully faithful. Finally, to prove that $H^0\mathcal{F}$ is essentially surjective we use that its essential image is a strictly (i.e. closed under isomorphism) full subcategory of $H^0\mathcal{B}^{\textit{pre-tr}}$ that contains all the objects of $\mathcal{B}$ and is closed under shifts and cones, so it has to coincide with all of $H^0\mathcal{B}^{\textit{pre-tr}}$. □

From this proposition follows that a dg-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if and only if $H^0\mathcal{F}\colon H^0\mathcal{A}^{\textit{pre-tr}} \to H^0\mathcal{B}^{\textit{pre-tr}}$ is an equivalence of categories. Similarly, recalling Lemma 3.70, we have

Proposition 3.79. *If $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a quasi equivalence, then*

$$\mathrm{Ind}_{\mathcal{F}}\colon \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B})$$

is a quasi-equivalence.

Proof. $\mathcal{SF}(\mathcal{A})$ and is pretriangulated, so it is sufficient to prove that the induced functor

$$H^0\,\mathrm{Ind}_{\mathcal{F}}\colon D(\mathcal{A}) \to D(\mathcal{B})$$

is an equivalence. Full faithfulness follows directly from Lemma 1.19: we know that $\operatorname{Ind}_{\mathcal{F}}$ carries representable $\mathcal{A}$-modules to representables $\mathcal{B}$-modules, and since $\mathcal{F}$ is a quasi-equivalence $H^0 \operatorname{Ind}_{\mathcal{F}}$ is fully faithful when restricted to the representable. We also know that those are compact generators for $D(\mathcal{A})$. Finally, one proves that $H^0 \operatorname{Ind}_{\mathcal{F}}$ is essentially surjective by using the fact the its essential image contains $\mathcal{B}$, and is closed under shifts, cones, coproducts and homotopy equivalences: since by Lemma 3.67 every module in $\mathcal{SF}(\mathcal{B})$ can be represented as the cone of a morphism between free $\mathcal{B}$-modules, this concludes the argument. □

Remark. In a similar way, one proves that if $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, so is the induced dg-functor

$$\operatorname{Ind}_{\mathcal{F}}\colon \text{h-proj}(\mathcal{A}) \to \text{h-proj}(\mathcal{B}).$$

Remark. We have just proved that, if $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then it induces an exact equivalence

$$D(\mathcal{A}) \cong H^0\mathcal{SF}(\mathcal{A}) \overset{\operatorname{Ind}_{\mathcal{F}}}{\to} H^0\mathcal{SF}(\mathcal{B}) \cong D(\mathcal{B}).$$

Since $\operatorname{Ind}_{\mathcal{F}}$ is adjoint to $\operatorname{Res}_{\mathcal{F}}$, this implies that the functor

$$D(\mathcal{B}) \to D(\mathcal{A})$$

induced by composition with $\mathcal{F}$ is an equivalence.

We now get to a fundamental definition.

Definition 3.80. The category **Hqe** is defined as the localization of $\mathbf{dgcat}_k$ at the quasi-equivalences.

Remark. Since the quasi-equivalences in particular induce equivalences at the level of the homotopy categories, a morphism $Q\colon \mathcal{A} \to \mathcal{B}$ in **Hqe** induces an exact functor $H^0Q\colon H^0\mathcal{A} \to H^0\mathcal{B}$, that is well defined up to equivalence; furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated, H^0Q is exact. To be more precise, denoting with $\mathbf{cat}_k$ the category of small k-linear categories, the functor $H^0\colon \mathbf{dgcat}_k \to \mathbf{cat}_k$ sends quasi-equivalences to equivalence of categories, so descends to a functor from **Hqe** to $\mathbf{cat}_k[\mathcal{W}^{-1}]$, where $\mathcal{W}$ is the class of (k-linear) equivalences. Fortunately, one can prove that this localization is easy to describe as an homotopy category: it has the same objects as $\mathbf{cat}_k$, and as morphisms isomorphism classes of k-linear functors. This will allow us to sweep this step under the rug, treating morphisms in $\mathbf{cat}_k[\mathcal{W}^{-1}]$ as genuine functors. Note that this description of the localization is in general not possible in $\mathbf{dgcat}_k$, since (for example) not all quasi-equivalences admit a quasi-inverse.

In the following we will see that **Hqe** is the correct category in which to discuss the uniqueness of enhancements of triangulated categories. The reasons are both conceptual - a quasi-equivalence between pretriangulated dg-categories is just a dg-functor inducing an isomorphism at the level of the homotopy category - and practical, since one can actually get some very strong uniqueness results in **Hqe**.

The structure of **Hqe** is very complex, and describing it in detail is outside of the scope of this thesis; however, it is still worth it to give some foundational results, that explain how the hom-spaces of **Hqe** are formed. A recollection of these results can be found in section 4 of [Kel06]. More detailed discussion are in [Toë07], [Tab05] and [Bel13]. We begin with a simplified form of a Theorem of Tabuada ([Tab05], or appendix B in [Bel13] for an english version).

Theorem 3.81. $\mathbf{dgcat}_k$ *admits the structure of a model category where the weak equivalences are the quasi-equivalences, in which all dg-categories are fibrant objects.*

As a consequence of this theorem, the category **Hqe** has small hom-sets, and every morphism in **Hqe** can be represented as a roof of dg-functors

$$\mathcal{A} \xleftarrow{\sim} \mathcal{A}_{cof} \to \mathcal{B},$$

where $\mathcal{A} \xleftarrow{\sim} \mathcal{A}_{cof}$ is a cofibrant replacement for $\mathcal{A}$.

We have already seen that the category $\mathbf{dgcat}_k$ is a monoidal category, possessing a well behaved tensor product: it makes sense to ask whether this structure descends to the localized category **Hqe**, defining a derived tensor product of dg-categories $\otimes^L$. The naive approach does not work, since the functor $\mathcal{A} \otimes -\colon \mathbf{dgcat}_k \to \mathbf{dgcat}_k$ does not preserve quasi-equivalences, so does not descend to a functor $\mathbf{Hqe} \to \mathbf{Hqe}$: however, as is often the case, if $\mathcal{A}$ is cofibrant (in the sense of the model structure of Theorem 3.81) then the functor $\mathcal{A} \otimes -$ does preserve quasi-equivalences; this allows us to define the derived tensor product $\mathcal{A} \otimes^L \mathcal{B}$ as $\mathcal{A}_{cof} \otimes \mathcal{B}$ where $\mathcal{A}_{cof}$ is a cofibrant replacement for $\mathcal{A}$, giving a well defined functor $\mathbf{Hqe} \times \mathbf{Hqe} \to \mathbf{Hqe}$. In general it is not necessary to consider cofibrant replacements, but it is sufficient to take h-flat (or k-flat in some texts) replacements (a dg-category $\mathcal{A}$ is h-flat if for all $A, B \in \mathcal{A}$ the functor $\mathcal{A}(X, Y) \otimes -$ preserves quasi-isomorphisms); this is easier since, for example, if k is a field all dg-categories are h-flat. Note that all cofibrant dg-categories are in particular h-flat.

Remark. It should be noted that, although every dg-category admits a cofibrant resolution, "most" dg-categories are not cofibrant, and cofibrant resolutions are not easy to compute in practice. On the other hand, h-flat categories are easier to come by, as already seen in the case of a field.

Remark. It follows from Theorem 3.81 that a morphism $Q\colon \mathcal{A} \to \mathcal{B}$ in **Hqe** between pretriangulated dg-categories is an isomorphism in **Hqe** if and only if the induced functor $H^0Q\colon H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence of triangulated categories.

There is a more natural way to understand morphisms in **Hqe**. Suppose that $\mathcal{A}$ is h-flat, otherwise replace $\mathcal{A}$ with a quasi-equivalent h-flat dg-category. Denote with $\underline{\mathcal{B}}$ the essential image of the derived Yoneda embedding $H^0\mathcal{B} \to D(\mathcal{B})$; that is, the full subcategory of $\mathcal{HB}$ (or of $D(\mathcal{B})$, or of $\mathcal{M}od\text{-}\mathcal{B}$) spanned by all objects quasi-isomorphic to representable $\mathcal{B}$-modules. Consider now a dg $\mathcal{A}$-$\mathcal{B}$ bimodule X; it induces (in fact, is the same thing as) a dg-functor

$$\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B};$$

Definition 3.82. An $\mathcal{A}$-$\mathcal{B}$ bimodule X is called right quasi-representable if the image of the induced functor

$$\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B};$$

is is contained in $\underline{\mathcal{B}}$, i.e. if $X(A,-) \in \underline{\mathcal{B}}$ for all $A \in \mathcal{A}$. Equivalently, X is right quasi-representable if

$$-\otimes^L X\colon D(\mathcal{A}) \to D(\mathcal{B})$$

carries representable $\mathcal{A}$-modules to objects isomorphic in $D(\mathcal{B})$ to representable $\mathcal{B}$-modules.

We denote with $\operatorname{rep}(\mathcal{A},\mathcal{B})$ the full the subcategory of $D(\mathcal{A}^{op}\otimes\mathcal{B})$ formed by all right quasi-representable bimodules, and with $\operatorname{rep}_{\mathrm{dg}}(\mathcal{A},\mathcal{B})$ the subcategory of $\mathcal{M}od\text{-}\mathcal{A}^{op}\otimes\mathcal{B}$ formed by h-projective quasi-representable dg-modules, i.e. a canonical dg-enhancement of $\operatorname{rep}(\mathcal{A},\mathcal{B})$.

Remark. There are several different but equivalent definitions of $\operatorname{rep}(\mathcal{A},\mathcal{B})$ in the literature. For example, in [CS15] a right quasi-representable $\mathcal{A}$-$\mathcal{B}$ bimodule is by definition h-projective: this is always possible, since the notion of being quasi-representable is invariant under quasi-isomorphism. The advantage of this approach comes from the fact that if X is h-projective, one can prove that the condition for $X(A,-)$ to be quasi-isomorphic to a representable module is equivalent to that of being homotopically equivalent to a representable module. As a consequence, the image of the morphism $\mathcal{A} \to \mathcal{M}od\text{-}\mathcal{B}$ is contained in the essential image of the derived Yoneda embedding $H^0\mathcal{B} \to \mathcal{HB}$, that we will denote $\hat{\mathcal{B}}$. The advantage of this approach is that $\mathcal{B} \hookrightarrow \hat{\mathcal{B}}$ is always a quasi equivalence, so X induces a morphism in **Hqe** given by the roof $\mathcal{A} \to \hat{\mathcal{B}} \overset{\sim}{\hookleftarrow} \mathcal{B}$. We will see in a few moments that all morphisms in **Hqe** in fact arise in this way.

The following theorem is originally due to Toën, as a corollary of a more general result regarding the simplicial localization of $\mathbf{dgcat}_k$ (see [Toë07]). A more elementary proof has been given by Canonaco and Stellari in [CS15]. We keep supposing $\mathcal{A}$ to be h-flat.

Theorem 3.83. *The set of morphisms in* **Hqe** *between two dg-categories* $\mathcal{A}$ *and* $\mathcal{B}$ *is in natural bijection with the isomorphism classes in* $\operatorname{rep}(\mathcal{A}, \mathcal{B})$.

Idea of the bijection. We have already seen in the remark above that a right quasi-representable h-projective $\mathcal{A}$-$\mathcal{B}$ bimodule defines a morphism in **Hqe** from $\mathcal{A}$ to $\mathcal{B}$. Vice versa, suppose we have a morphism $f \in \operatorname{Hom}_{\mathbf{Hqe}}(\mathcal{A}, \mathcal{B})$; write f as a composition $\mathcal{A} \xleftarrow{I} \mathcal{A}' \xrightarrow{g} \mathcal{B}$, that we can consider a formal composition $g \circ I^{-1}$, where I is a quasi-equivalence and g is a dg-functor. Since g is a dg-functor $\mathcal{C} \to \mathcal{B}$, we can compose it with the dg-Yoneda embedding to find a (right quasi-representable) $\mathcal{A}'$-$\mathcal{B}$-bimodule G. Similarly, we can find a right quasi-representable $\mathcal{A}'$-$\mathcal{A}$-bimodule J associated to I. Now one has to use the fact that I is a quasi-equivalence to find an "inverse" J^{-1} to J; once this is done, we will find the bimodule F corresponding to f by setting $F = J^{-1} \otimes_{\mathcal{B}} G$. In order to find J^{-1}, one first proves that since I is a quasi-equivalence it induces a bijection

$$\operatorname{rep}(\mathcal{A}, \mathcal{A}') \to \operatorname{rep}(\mathcal{A}, \mathcal{A});$$

finally, J^{-1} can be defined as the preimage via this bijection of the "diagonal" $\mathcal{A}$-$\mathcal{A}$ bimodule $\mathcal{A}(-,-)$. For more details on this approach, the reader can consult [CS15, Proposition 3.12]. □

From now on, we will call morphisms in **Hqe** quasi-functors.

Remark. Some authors call quasi-functors the bimodules in $\operatorname{rep}(\mathcal{A}, \mathcal{B})$; according to the convention that we will follow, a quasi-functor is an isomorphism class of bimodules.

One should now ask whether the monoidal category $(\mathbf{Hqe}, \otimes^L)$ is closed , possessing an internal hom. The internal hom of $\mathbf{dgcat}_k$ does not work in this case: the dg-functor $\mathcal{H}om(\mathcal{A}, -)$ does not preserve quasi-equivalences even when $\mathcal{A}$ is cofibrant. The correct solution applies the constructions made in the above paragraph: the following theorem is also from [Toë07], with a more elementary proof present in [CS15].

Theorem 3.84. *The monoidal category* $(\mathbf{Hqe}, \otimes^L)$ *admits an internal hom given by the dg category*

$$\mathcal{RH}om(\mathcal{A}, \mathcal{B}) = \operatorname{rep}_{dg}(\mathcal{A}_{cof}, \mathcal{B})$$

where $\mathcal{A}_{cof}$ is a cofibrant (or more in general h-flat) replacement for $\mathcal{A}$. That is, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Hqe}}(\mathcal{A} \otimes^L \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_{\mathbf{Hqe}}(\mathcal{A}, \mathcal{RH}om(\mathcal{B}, \mathcal{C}))$$

for $\mathcal{A}, \mathcal{B}, \mathcal{C} \in$ **Hqe**.

3.7 Drinfeld's dg-quotient

Recall that, given a triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ of a triangulated category, the Verdier quotient $\mathcal{T}/\mathcal{S}$ was characterized by the fact that any functor $\mathcal{T} \to \mathcal{T}'$ annihilating $\mathcal{S}$ would factor through $\mathcal{T}/\mathcal{S}$. A similar construction exists for dg-categories, called the Drinfeld dg-quotient.

Definition 3.85. An object A in a dg-category $\mathcal{A}$ is said to be contractible if the identity $\mathrm{id}_A \in \mathcal{A}(A, A)^0$ admits a nullhomotopy (i.e. is a boundary).

Remark. if $\mathcal{A}$ is pretriangulated (in fact, it is enough for it to contain a zero object in its homotopy category) being contractible is equivalent to being isomorphic to 0 in $H^0\mathcal{A}$.

Let now $\mathcal{A}$ be a dg-category, and $\mathcal{B} \subseteq \mathcal{A}$ a full dg-subcategory. We say that a quasi-functor $F \colon \mathcal{A} \to \mathcal{C}$ annihilates $\mathcal{B}$ if $H^0F \colon H^0\mathcal{A} \to H^0\mathcal{C}$ sends all the objects of $\mathcal{B}$ to contractible objects. Note that this definition does not depend on the chosen representative of $H^0\mathcal{F}$.

Definition 3.86. The Drinfeld quotient $\mathcal{A}/\mathcal{B}$ is a dg-category $\mathcal{A}/\mathcal{B}$ equipped with a quasi-functor $Q \colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ satisfying the following universal property: any morphism $\mathcal{A} \xrightarrow{F} \mathcal{B}$ in **Hqe** annihilating $\mathcal{B}$ factors uniquely through Q, as shown in the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle Q}\downarrow & \nearrow & \\ \mathcal{A}/\mathcal{B}. & & \end{array}$$

It follows from the definition that the Drinfeld quotient, if exists, is unique up to a unique isomorphism in **Hqe**. The uniqueness of the Drinfeld quotient can actually be stated in a much more precise way; this has been done both in the original paper [Dri04], and in more detail in [Tab10]; in particular, the factorization can be seen both at the level of $\mathrm{rep}(\mathcal{A}, \mathcal{C})$ and at that of the dg-enhancement $\mathrm{rep}_{\mathrm{dg}}(\mathcal{A}, \mathcal{C})$. We will not need those more precise characterizations so we omit them, referring the interested reader to [Tab10].

Theorem 3.87 ([Dri04]). *Let $\mathcal{A}$ be a small dg-category, and $\mathcal{B} \subseteq \mathcal{A}$ a full dg-subcategory. Then the Drinfeld quotient $\mathcal{A}/\mathcal{B}$ exists, and*

$$H^0Q\colon H^0\mathcal{A} \to H^0\left(\mathcal{A}/\mathcal{B}\right)$$

is essentially surjective. Furtermore if $\mathcal{A}$ is h-flat (so for example is k is a field), Q can be taken to be a dg-functor (i.e. a zig-zag of length one).

The construction is as follows: first take a h-flat resolution of $\mathcal{A}$, $\tilde{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}$. Then consider the subcategory $\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{A}}$ corresponding to $\mathcal{B}$ under the resolution, and for every object $B \in \tilde{\mathcal{B}}$, add to $\tilde{\mathcal{A}}$ a contracting homotopy, i.e. a morphism h_B of degree -1 such that $d(h_B) = \mathrm{id}_B$: the dg-category obtained in this way is the quotient $\mathcal{A}/\mathcal{B}$, and the morphism Q is the composition of the inverse (in **Hqe**) of the resolution with the natural dg-functor $\tilde{\mathcal{A}} \to \mathcal{A}/\mathcal{B}$, that is the identity on the objects. Of course, if $\mathcal{A}$ is already h-flat one takes Q to be the natural dg-functor, without taking the resolution.

Remark. There is another, completely analogous way to describe $\mathcal{A}/\mathcal{B}$ in terms of the model structure on $\mathbf{dgcat}_k$; this was done in [Tab10]. Denote with $\mathcal{S}$ the dg-category with two objects, called 1 and 2, and with hom-spaces

$$\mathcal{S}(1,1) = k, \;\; \mathcal{S}(2,2) = k, \;\; \mathcal{S}(1,2) = k \;\text{ and }\; \mathcal{S}(2,1) = 0,$$

where k is considered as a chain complex concentrated in degree 0, and composition is given by multiplication. This is the dg-equivalent of the walking arrow category, in the sense that a dg-functor $\mathcal{S} \to \mathcal{A}$ is just the specification of a closed, degree 0 morphism in $\mathcal{A}$. Now let $\mathcal{D}$ be the dg-category with two objects, again denoted 1 and 2,

$$\mathcal{D}(1,1) = k, \;\; \mathcal{D}(2,2) = k, \;\; \mathcal{S}(2,1) = 0$$

and $\mathcal{D}(1,2)$ equal to the complex

$$\cdots \to 0 \to k \xrightarrow{\mathrm{id}} k \to 0 \to \cdots \tag{3.21}$$

where k is located in degree -1 and 0; composition is again induced by multiplication. In this case, a dg-functor $\mathcal{D} \to \mathcal{A}$ is the datum of a closed degree 0 morphism f together with a nullhomotopy of f. There is a natural dg-functor $\mathcal{S} \to \mathcal{D}$ corresponding to the inclusion of k in the degree 0 of (3.21). Suppose now $\mathcal{A}$ to be cofibrant (otherwise take a resolution) and let $\mathcal{B} \subseteq \mathcal{A}$ be a full dg-subcategory. Recall that, since $\mathbf{dgcat}_k$ admits the structure of a model category, in particular it admits all limits and colimits. Define now $\mathcal{A}/\mathcal{B}$ via the following pushout diagram in $\mathbf{dgcat}_k$

$$\begin{tikzcd} \coprod_{x\in\mathcal{B}}\mathcal{S} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ \coprod_{x\in\mathcal{B}}\mathcal{D} \arrow[r] & \mathcal{A}/\mathcal{B} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where the upper horizontal arrow corresponds to the specification of the identities of the objects of $\mathcal{B}$, and the left vertical arrow to the natural dg-functor $\mathcal{S} \to \mathcal{D}$. Suppose now that we have a dg-functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{C}$ that annihilates $\mathcal{B}$. This induces a dg-functor $\coprod_{x\in\mathcal{B}}\mathcal{D} \to \mathcal{C}$, corresponding to the nullhomotopies of $\mathrm{id}_{\mathcal{F}B}$, for $B \in \mathcal{B}$, making the diagram

$$\begin{tikzcd} \coprod_{x\in\mathcal{B}}\mathcal{S} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \arrow[ddr, bend left] & \\ \coprod_{x\in\mathcal{B}}\mathcal{D} \arrow[r] \arrow[drr, bend right] & \mathcal{A}/\mathcal{B} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\ & & \mathcal{C} \end{tikzcd}$$

commutative. Then the universal property of the pushout provides a dg-functor $\mathcal{A}/\mathcal{B} \to \mathcal{C}$ factoring $\mathcal{F}$. In [Tab10], the author then uses the model structure on $\mathbf{dgcat}_k$ to show that this construction extends to $\mathbf{Hqe}$, and that there the correspondence between quasi-functors $\mathcal{A}/\mathcal{B} \to \mathcal{C}$ and quasi-functors $\mathcal{A} \to \mathcal{C}$ annihilating $\mathcal{B}$ is bijective.

One of the main features of dg-quotients that they provide enhancements to Verdier quotients: we will now explore precisely how. For any dg-category $\mathcal{A}$, denote $H^0\mathcal{A}^{\text{pre-tr}}$ with $\mathcal{A}^{tr}$; clearly $\mathcal{A}^{tr}$ is triangulated, and if $\mathcal{A}$ is pretriangulated $\mathcal{A}^{tr}$ is equivalent to $H^0\mathcal{A}$. Consider the quasi-functor $Q\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$; as in the case of a dg-functor, it induces a quasi-functor

$$Q^{\text{pre-tr}}\colon \mathcal{A}^{\text{pre-tr}} \to \left(\mathcal{A}/\mathcal{B}\right)^{\text{pre-tr}}$$

and therefore an exact functor

$$Q^{tr}\colon \mathcal{A}^{tr} \to \left(\mathcal{A}/\mathcal{B}\right)^{tr}$$

that annihilates (in the sense of triangulated categories) $\mathcal{B}^{tr} \subseteq \mathcal{A}^{tr}$, where $\mathcal{B}^{tr}$ is the homotopy category of the closure of $\mathcal{B}$ under shifts and cones. By definition of Verdier quotient, Q^{tr} induces a functor

$$\Phi\colon \mathcal{A}^{tr}/\mathcal{B}^{tr} \to \left(\mathcal{A}/\mathcal{B}\right)^{tr}.$$

Theorem 3.88 ([Dri04]). *The functor*

$$\Phi \colon \mathcal{A}^{tr}/\mathcal{B}^{tr} \to \left(\mathcal{A}/\mathcal{B}\right)^{tr}.$$

is an equivalence.

It should be noted that this is equivalent to saying that the quasi-functor $Q^{pre\text{-}tr} \colon \mathcal{A}^{pre\text{-}tr} \to \left(\mathcal{A}/\mathcal{B}\right)^{pre\text{-}tr}$ induces in homotopy the usual Verdier localization functor (up to equivalence).

Corollary 3.89. *Suppose $\mathcal{A}$ is pretriangulated. Then any quotient $\mathcal{A}/\mathcal{B}$ is pretriangulated. In particular, if $\mathcal{B}$ is also pretriangulated, $\mathcal{A}/\mathcal{B}$ is an enhancement of $H^0\mathcal{A}/H^0\mathcal{B}$.*

Proof. We have to prove that the natural inclusion

$$H^0\left(\mathcal{A}/\mathcal{B}\right) \hookrightarrow \left(\mathcal{A}/\mathcal{B}\right)^{tr}$$

is essentially surjective. By definition of $Q^{pre\text{-}tr}$, the diagram

$$\begin{array}{ccc} \mathcal{A} & \hookrightarrow & \mathcal{A}^{pre\text{-}tr} \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q^{pre\text{-}tr}} \\ \mathcal{A}/\mathcal{B} & \longrightarrow & \left(\mathcal{A}/\mathcal{B}\right)^{pre\text{-}tr} \end{array}$$

is commutative. This induces the diagram

$$\begin{array}{ccc} H^0\mathcal{A} & \hookrightarrow & \mathcal{A}^{tr} \\ \downarrow{\scriptstyle H^0Q} & & \downarrow{\scriptstyle Q^{tr}} \\ H^0\left(\mathcal{A}/\mathcal{B}\right) & \longrightarrow & \left(\mathcal{A}/\mathcal{B}\right)^{tr}. \end{array}$$

By the remark below Theorem 3.88, Q^{tr} is essentially surjective. Since the upper arrow is essentially surjective by hypothesis (being $\mathcal{A}$ pretriangulated), it follows that the composition

$$H^0\mathcal{A} \overset{H^0Q}{\to} H^0\left(\mathcal{A}/\mathcal{B}\right) \hookrightarrow \left(\mathcal{A}/\mathcal{B}\right)^{tr}$$

is essentially surjective, therefore the inclusion $H^0\mathcal{A} \overset{H^0Q}{\to} H^0\left(\mathcal{A}/\mathcal{B}\right)$ is essentially surjective. □

Example 3.11. If $\mathcal{A}$ is a dg-category, the dg-category ${\mathcal{M}od\text{-}\mathcal{A}}/{\mathrm{Ac}(\mathcal{A})}$ is an enhancement to $D(\mathcal{A})$. Again by Theorem 3.88, the composition

$$\mathcal{SF}(A) \hookrightarrow \mathcal{M}od\text{-}\mathcal{A} \to {\mathcal{M}od\text{-}\mathcal{A}}/{\mathrm{Ac}(\mathcal{A})}$$

is a quasi-equivalence: indeed, the composition of the inclusion

$$H^0\mathcal{SF}(\mathcal{A}) \to \mathcal{HA}$$

with the localization $\mathcal{HA} \to D(\mathcal{A})$ is an equivalence of categories. In the following we will often make no distinction between $D(\mathcal{A})$ and $H^0\mathcal{SF}(\mathcal{A})$, leaving implicit the equivalence.

3.8 Notes for Chapter 3

Dg-categories appeared fist in the work of Kelly in the context of enriched category theory, but were not studied in depth until the nineties, with the works of Keller ([Kel94]) and Drinfeld ([Dri04]). More recently, Tabuada ([Tab05], [Tab10]) and Toën ([Toë07]) have successfully applied methods from homotopy theory and model categories to develop the homotopy theory of dg-categories. The idea of dg-categories as enhancements of triangulated categories was suggested by Bondal and Kapranov in [BK91]. A through overview of the theory of dg-categories is Keller's ICM address [Kel06].

Chapter 4

Uniqueness of enhancements

We are finally ready to discuss the central part of this thesis, the uniqueness (or lack of thereof) of a dg-enhancement of an algebraic triangulated category. As was already discussed, it is not entirely obvious what it means for a triangulated category to have a unique enhancement. After having introduced the category **Hqe**, we can however give the following definition:

Definition 4.1. Let $\mathcal{T}$ be a k-linear triangulated category. We say that $\mathcal{T}$ admits a unique enhancement if, given two different dg-enhancements $(\mathcal{A}, \epsilon)$ and $(\mathcal{B}, \eta)$, $\mathcal{A}$ is isomorphic to $\mathcal{B}$ in **Hqe**.

There exist other (stronger) definitions of uniqueness of an enhancement, but we will focus on this one. It should be noticed that from a theoretical perspective this definition is not optimal: after all, the main theme of this thesis is that when localizing, it makes sense to keep track of "higher morphisms" between objects. On the contrary, our definition of uniqueness is nothing but a statement about uniqueness in the localized category **Hqe**, where we have already forgotten about most of the structure. It is nonetheless a challenging question to understand what "higher morphisms" are in $\mathbf{dgcat}_k$; the approach taken for example in [Toë07], is to consider the Dwyer-Kan localization of $\mathbf{dgcat}_k$ at the quasi-equivalences. We will not need this this nuance though, so we will keep dealing with the "localized" category **Hqe**.

The hope that any algebraic triangulated category admits a unique dg-enhancement is quickly proven to be vain.

Example 4.1. Let p be a prime number and let $\mathbb{F}_p$ be the field with p elements. Call $\mathbf{T}_p$ the category of vector spaces over $\mathbb{F}_p$. It admits a triangulated structure, with translation given by the identity functor and as distinguished triangles all the triangles

$$X \to Y \to Z \to X$$

such that the sequence

$$\cdots \to Z \to X \to Y \to Z \to X \to Y \to Z \to \cdots$$

is exact. Let now $R_1 = \mathbb{Z}/p^2\mathbb{Z}$ and $R_2 = \mathbb{F}_p[x]/(x^2)$. Denote with $\mathbf{C}^{ai}_{\mathrm{dg}}(R_j\text{-Mod})$ the full dg-subcategory of $\mathbf{C}_{\mathrm{dg}}(R_j\text{-Mod})$ with objects the acyclic complexes with injective components, for $j = 1, 2$. It can be proved (see for example [CS17, Section 3.3]) that $\mathbf{C}^{ai}_{\mathrm{dg}}(R_1\text{-Mod})$ and $\mathbf{C}^{ai}_{\mathrm{dg}}(R_2\text{-Mod})$ are both dg-enhancements of the category $\mathbf{T}_p$, but are not isomorphic in **Hqe**.

Remark. In the example above, we have considered the category $\mathbf{T}_p$ only as a $\mathbb{Z}$-linear category; if we want to keep track of the $\mathbb{F}_p$-linearity, we lose the enhancement $\mathbf{C}^{ai}_{\mathrm{dg}}(R_1\text{-Mod})$, since this dg-category is not $\mathbb{F}_p$-linear. This leads to the natural question: are there triangulated categories, linear over a field k, with different non-equivalent k-linear dg-enhancement? A positive answer has been given by Rizzardo and Van den Bergh, who gave in [RB21] such an example.

Nonetheless, it has been proven that many (in fact, most) interesting algebraic triangulated categories admit a unique enhancement: in the last fifteen years several results of increasing generality have come out, culminating in the very recent [CNS21]. Here we will explore in detail the first of those to appear, [LO10], discussing the others in the last chapter.

4.1 Overview of the results

In this section we will suppose k to be a field.

Theorem 4.2 ([LO10])**.** *Let $\mathcal{A}$ be a small k-linear category, considered as a dg-category concentrated in degree 0. Let $L \subseteq D(\mathcal{A})$ be a localizing subcategory. Assume that the following conditions hold:*

a) *The quotient functor $\pi\colon D(\mathcal{A}) \to D(\mathcal{A})/L$ admits a right adjoint;*

b) *For every $Y \in A$, the object $\pi(h_Y)$ is compact in $D(A)/L$;*

c) *For every $Y, Z \in \mathcal{A}$ and for any $i < 0$, we have*

$$\mathrm{Hom}_{D(\mathcal{A})/L}(\pi(h_Y), \pi(h_Z)[i]) = 0.$$

Then the quotient $D(\mathcal{A})/L$ admits a unique enhancement.

Remark. Since $D(\mathcal{A})$ is compactly generated, if we suppose that $D(\mathcal{A})/L$ has small hom-sets, then by Proposition 1.33 the functor μ exists automatically.

Remark. If $\mathcal{A}$ is an abelian category, in general the derived category of $\mathcal{A}$ seen as dg-category is different from the usual derived category of $\mathcal{A}$. An element of the first is a chain complex of functors from $\mathcal{A}^{op}$ to the category of k-modules, while an object of the second is a chain complex of elements of $\mathcal{A}$. Nonetheless those do have a relationship, that we will briefly explore later. Unfortunately it is customary to denote both objects as $D(\mathcal{A})$; when there is ambiguity, we will refer to the dg-version as the dg-derived category and to the abelian one as simply the derived category .

We will see the proof of this theorem in the following sections: now, we look at some of his consequences. We will need to briefly introduce some facts about the localizations of abelian categories: a reference for all of them is [Gar16], together with section 7 of [LO10].

Let $\mathcal{A}$ be a small k-linear category. We can consider the abelian category

$$\mathrm{Mod}\text{-}\mathcal{A} = \mathrm{Fun}_k(\mathcal{A}, k\text{-}\mathrm{Mod})$$

of k-linear functors from $\mathcal{A}$ to the category of k-modules. If we consider $\mathcal{A}$ as a dg-category concentrated in degree 0, we find that the dg-category of dg-modules $\mathcal{M}od\text{-}\mathcal{A}$ coincides with $\mathbf{C}_{\mathrm{dg}}(\mathrm{Mod}\text{-}\mathcal{A})$, therefore the dg-derived category $D(\mathcal{A})$ coincides with the (classical) derived category $D(\mathrm{Mod}\text{-}\mathcal{A})$.

Definition 4.3. Let $\mathcal{C}$ be an abelian category. We say that $\mathcal{C}$ is a Grothendieck abelian category, or Grothendieck category, if the following hold:

- $\mathcal{C}$ admits all small colimits;
- Direct limits in $\mathcal{C}$ of short exact sequences are exact: for any directed set I and short exact sequences

  $$0 \to A_i \to B_i \to C_i \to 0$$

  with $i \in I$, the induced sequence

  $$0 \to \varinjlim_i A_i \to \varinjlim_i B_i \to \varinjlim_i C_i \to 0$$

  is exact;
- There exists a set $\mathcal{S} \subseteq \mathrm{Ob}(\mathcal{C})$ of generators of $\mathcal{C}$, in the sense that for any object $C \in \mathcal{C}$ there exists an epimorphism $S \twoheadrightarrow \mathcal{C}$, where S is a direct sum of objects of $\mathcal{S}$.

Note that we can always suppose for $\mathcal{C}$ to admit a single generator, by taking the direct sum of all the elements in $\mathcal{S}$. As a consequence of the Gabriel-Popescu theorem ([GP64]) and of [LO10, Lemma 7.2] we have the following proposition.

Proposition 4.4. *Let $\mathcal{C}$ be a Grothendieck category that admits a set of generators $\mathcal{A}$ that are compact in the derived category $D(\mathcal{C})$. Then there exists a localizing subcategory $\mathcal{N} \subseteq D(\mathcal{A})$ and an equivalence of categories*

$$D(\mathcal{C}) \xrightarrow{\sim} {D(\mathrm{Mod}\text{-}\mathcal{A})}\big/{\mathcal{N}}$$

where the derived category on the right is to be intended in the dg-sense. Furthermore, the quotient

$$D(\mathrm{Mod}\text{-}\mathcal{A}) \to {D(\mathrm{Mod}\text{-}\mathcal{A})}\big/{\mathcal{N}}$$

satisfies the hypotheses of Theorem 4.2.

Corollary 4.5. *Let $\mathcal{A}$ be a Grothendieck abelian category such that the derived category $D(\mathcal{C})$ has small hom-sets. Assume further that $\mathcal{C}$ has a set of generators that are compact in the derived category $D(\mathcal{C})$. Then $D(\mathcal{C})$ admits a unique enhancement.*

This has some interesting geometric consequences.

Corollary 4.6. *Let X be a quasi-compact and separated scheme that has enough locally free sheaves. Then the derived category of quasi-coherent sheaves $D(\mathrm{Qcoh}\, X)$ admits a unique enhancement.*

4.2 Outline of the proof of Theorem 4.2

In this section we give an outline of the main steps of the proof; this will hopefully help the reader navigate through the next sections. We also fix some notation and make some useful constructions. We begin with an important remark; for how we have proceeded, all of our objects constructions must live in the category $\mathbf{dgcat}_k$, the category of small dg-categories. Nonetheless, often we will work with categories that are not small: namely, the category $\mathcal{M}od\text{-}\mathcal{A}$ is not small even when $\mathcal{A}$ is. This is a delicate issue, but we will not discuss it here, working with arbitrary categories as if they were small. A through discussion, and a way to solve this issue, is Appendix A of [LO10]. Let us now get to the actual proof.

To begin with, observe that the category ${D(\mathcal{A})}/{L}$ has an automatic: denoting with $\mathcal{L}$ the full dg-subcategory having the same objects as L, the Drinfeld quotient ${\mathcal{M}od\text{-}\mathcal{A}}/{\mathcal{L}}$ is an enhancement of ${D(\mathcal{A})}/{L}$. By Proposition 1.29, ${\mathcal{SF}(\mathcal{A})}/{\mathcal{L} \cap \mathcal{SF}(\mathcal{A})}$ is another, equivalent, enhancement. Suppose now that ${D(\mathcal{A})}/{L}$ has another enhancement $(\mathcal{C}, \epsilon)$. In order to prove the theorem, it would then suffice to find an isomorphism in $\mathbf{Hqe}$ between $\mathcal{C}$ and ${\mathcal{SF}(\mathcal{A})}/{\mathcal{L} \cap \mathcal{SF}(\mathcal{A})}$. This is done in roughly four steps.

Step 1

The first step is to replace the category $\mathcal{C}$ with an equivalent, more manageable category: to begin with, since $\mathcal{A}$ in concentrated in degree 0, we have $\mathcal{A} = Z^0\mathcal{A} = H^0\mathcal{A}$. Therefore there is a natural sequence of dg-functors

$$\mathcal{A} = H^0(\mathcal{A}) \to D(\mathcal{A}) \xrightarrow{\pi} D(\mathcal{A})/L \xrightarrow{\epsilon} H^0\mathcal{C}. \tag{4.1}$$

Denote with $\mathcal{B} \subseteq \mathcal{C}$ the full dg-subcategory of $\mathcal{C}$ corresponding to the image of $\mathcal{A}$ via this sequence of functors. Notice that, since those are all injective on the objects, the objects of $\mathcal{B}$ are in bijection with those of $\mathcal{A}$. At this point, we will prove (using the hypotheses of Theorem 4.2) that there is an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{SF}(\mathcal{B})$ in **Hqe**. As a consequence, in order to prove Theorem 4.2 it will be sufficient to prove that there exists an isomorphism $\mathcal{SF}(\mathcal{A})/\mathcal{L} \cap \mathcal{SF}(\mathcal{A}) \xrightarrow{\sim} \mathcal{SF}(\mathcal{B})$ in **Hqe**. For future use, note also that by definition of $\mathcal{B}$, the composition (4.1) factors through the inclusion $H^0\mathcal{B} \hookrightarrow H^0\mathcal{C}$, hence defining a dg-functor $a\colon \mathcal{A} \to H^0\mathcal{B}$.

Step 2

In order to find the isomorphism $\mathcal{SF}(\mathcal{A})/\mathcal{L} \cap \mathcal{SF}(\mathcal{A}) \xrightarrow{\sim} \mathcal{SF}(\mathcal{B})$, we define a suitable quasi-functor $\mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B})$, and then use the universal property of the Drinfeld quotient to prove that it factors through the quotient $\mathcal{SF}(\mathcal{A})/\mathcal{L} \cap \mathcal{SF}(\mathcal{A})$. With this goal in mind, we need to briefly recall a construction: if C is a chain complex, we can define the truncation $\tau_{\leq 0}C$ as the complex given by

$$\cdots \to C^{-2} \to C^{-1} \to \operatorname{Ker} d_0 \to 0 \to \cdots$$

There is a natural chain map $i\colon \tau_{\leq 0}C \to C$; moreover, the projection $\operatorname{Ker} d_0 \to H^0C$ induces a chain map from $\tau_{\leq 0}C$ to H^0C considered as a chain complex concentrated in degree 0. Following this construction, and taking the dg-category $\mathcal{B}$ as in Step 1, we can define the dg-category $\tau_{\leq 0}\mathcal{B}$ by taking the objects to be the same of $\mathcal{B}$ and hom-spaces as the truncations of those of $\mathcal{B}$. We have then a natural dg-functor $p\colon \tau_{\leq 0}\mathcal{B} \xrightarrow{i} H^0\mathcal{B}$, where $H^0\mathcal{B}$ is seen as a dg-category concentrated in degree 0. In the hypotheses of Theorem 4.2, one can prove that p is a quasi-equivalence. There is a zig-zag of dg-functors

$$A \xrightarrow{a} H^0\mathcal{B} \xleftarrow{p} \tau_{\leq 0}\mathcal{B} \xrightarrow{i} \mathcal{B}$$

that induces the zig-zag

$$\mathcal{SF}(A) \xrightarrow{\operatorname{Ind}_a} \mathcal{SF}(H^0\mathcal{B}) \xleftarrow{\operatorname{Ind}_p} \mathcal{SF}(\tau_{\leq 0}\mathcal{B}) \xrightarrow{\operatorname{Ind}_i} \mathcal{SF}(\mathcal{B}).$$

Since p is a quasi-equivalence, so is Ind_p; therefore, the second zigzag defines a quasi-functor $\rho\colon \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B})$.

Step 3

Here, one has to prove that quasi-functor ρ annihilates $\mathcal{L} \cap \mathcal{SF}(\mathcal{A})$, therefore defining a quasi-functor ${}^{\mathcal{SF}(\mathcal{A})}\!/\!_{\mathcal{L} \cap \mathcal{SF}(\mathcal{A})} \to \mathcal{SF}(\mathcal{B})$. Thanks to Theorem 3.88, this reduces to proving that the corresponding functor between the homotopy categories annihilates $L \cap D(\mathcal{A})$; this is strictly a question about triangulated categories. The proof of this fact is one of the main technical burdens of the whole argument.

Step 4

To conclude the proof, we have to show that the quasi-functor

$$ {}^{\mathcal{SF}(\mathcal{A})}\!/\!_{\mathcal{L} \cap \mathcal{SF}(\mathcal{A})} \to \mathcal{SF}(\mathcal{B}) $$

is an isomorphism in **Hqe**. For this, it is sufficient to prove that the induced functor between the homotopy categories is an equivalence. Again, this is an issue exclusively about the triangulated side of the question, and again it is a bit delicate. Nonetheless, applying techniques similar to those of step 3, one can conclude this step and with it the proof.

4.3 Proof of Theorem 4.2

For the sake of brevity, we will not repeat the constructions done in the overview of the proof, but only refer to those.

Proof of Step 1

We have to find an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathcal{SF}(\mathcal{B})$ in **Hqe**. This is easily done: first observe that, since $\mathcal{B} \subseteq \mathcal{C}$, there is a natural dg-functor

$$ \begin{aligned} \Phi\colon \mathcal{C} &\to \mathcal{M}od\text{-}\mathcal{B} \\ C &\to \mathcal{B}(-, P) \end{aligned} $$

that is the composition of the dg-Yoneda embedding $\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{B}$ with the restriction along the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$. We can then compose this with the quotient and find a quasi-functor

$$ \mathcal{C} \to {}^{\mathcal{M}od\text{-}\mathcal{B}}\!/\!_{\mathrm{Ac}(\mathcal{B})}. $$

Since we know that
$$\mathcal{SF}(\mathcal{B}) \to {}^{\mathcal{M}od\text{-}\mathcal{B}}\!\big/_{\mathrm{Ac}(\mathcal{B})}$$
is a quasi-equivalence, this defines a quasi-functor $\mathcal{C} \to \mathcal{SF}(\mathcal{B})$ given by the roof
$$\mathcal{C} \to {}^{\mathcal{M}od\text{-}\mathcal{B}}\!\big/_{\mathrm{Ac}(\mathcal{B})} \xleftarrow{\sim} \mathcal{SF}(\mathcal{B}).$$
We now have to prove that this is an isomorphism: clearly, it is sufficient to prove that the dg-functor
$$\mathcal{C} \to {}^{\mathcal{M}od\text{-}\mathcal{B}}\!\big/_{\mathrm{Ac}(\mathcal{B})}$$
is a quasi equivalence: this will follow from the next lemmas.

Lemma 4.7. *Let $\mathcal{B} \subseteq \mathcal{C}$, and let*
$$\varphi\colon \mathcal{C} \to {}^{\mathcal{M}od\text{-}\mathcal{B}}\!\big/_{\mathrm{Ac}(\mathcal{B})}$$
as above. Assume that all the objects of $\mathcal{B}$ are compact in $H^0\mathcal{C}$. Then the induced functor
$$H^0\varphi\colon H^0\mathcal{C} \to D(\mathcal{B})$$
preserves coproducts existing in $H^0\mathcal{C}$.

Proof. Recall that $H^0\varphi$ is the composition of the map $H^0\Phi\colon H^0\mathcal{C} \to \mathcal{HB}$ with the quotient $\mathcal{HB} \to D(\mathcal{B})$, and the quotient is the identity on objects. Let now $\{C_i\}$ be a set of objects of $\mathcal{C}$, and suppose that the direct sum $\bigoplus_i C_i$ exists in $H^0\mathcal{C}$. Take $B \in \mathcal{B}$. Since B is compact in $H^0\mathcal{C}$, we have natural isomorphisms
$$\begin{aligned}&\mathrm{Hom}_{D(\mathcal{B})}(h_B, \varphi(\bigoplus_i C_i)) \cong \mathrm{Hom}_{\mathcal{HB}}(h_B, \Phi(\bigoplus_i C_i)) \cong H^0(\Phi(\bigoplus_i C_i)(B)) \cong\\ &\cong H^0\mathcal{C}(B, \bigoplus_i C_i) \cong \bigoplus_i H^0\mathcal{C}(B, C_i) \cong \bigoplus_i H^0(\Phi(C_i)(B)) \cong\\ &\cong H^0\big(\bigoplus_i \Phi(C_i)(B)\big) \cong \mathrm{Hom}_{\mathcal{HB}}(h_B, \bigoplus_i \Phi(C_i)) \cong \mathrm{Hom}_{D(\mathcal{B})}(h_B, \bigoplus_i \varphi(C_i)).\end{aligned}$$
Since $H^0\mathcal{B}$ compactly generates $D(\mathcal{B})$, Lemma 1.18 and the remark above guarantee that
$$\varphi(\bigoplus_i C_i) \cong \bigoplus_i \varphi(C_i)$$
in $D(\mathcal{B})$. □

Remark. By definition of φ, the composition

$$H^0\mathcal{B} \hookrightarrow H^0\mathcal{C} \stackrel{H^0\varphi}{\to} D(\mathcal{B})$$

coincides with the derived Yoneda embedding

$$H^0\mathcal{B} \hookrightarrow D(\mathcal{B}).$$

Lemma 4.8. *Let $\mathcal{B} \subseteq \mathcal{C}$, and let*

$$\varphi\colon \mathcal{C} \to {\mathcal{M}od\text{-}\mathcal{B}}/{\mathrm{Ac}(\mathcal{B})}$$

as above. Assume that $H^0\mathcal{C}$ contains arbitrary direct sums and $\mathrm{Ob}(\mathcal{B})$ *forms a set of compact generators for $H^0\mathcal{C}$. Then the induced functor*

$$H^0\varphi\colon H^0\mathcal{C} \to D(\mathcal{B})$$

is an equivalence of categories. As a consequence, the induced quasi-functor

$$H^0\mathcal{C} \to \mathcal{SF}(\mathcal{B})$$

is an isomorphism.

Proof. Since $\mathcal{B}$ compactly generates $H^0\mathcal{C}$, Lemma 1.19 implies that $H^0\varphi$ is fully faithful. Recall now that any element of $D(\mathcal{B})$ is isomorphic to a semi-free $\mathcal{B}$ modules and any semi-free module can be realized as the cone of a morphism between coproducts of (images via the Yoneda embedding) shifts of elements of $\mathcal{B}$. Therefore, since $H^0\varphi$ preserves representables, shifts, cones and coproducts, it is essentially surjective. □

In our case, since $L \subseteq D(\mathcal{A})$ is a localizing subcategory, $D(\mathcal{A}) \cong H^0\mathcal{C}$ admits arbitrary direct sums. Moreover, by Proposition 1.33 the objects of $\mathcal{B}$ compactly generate $H^0\mathcal{C} \cong {D(\mathcal{A})}/{L}$. This concludes the proof of step 1.

Proof of step 2

In this step, most of the work was already done in the overview. It only remains to prove the following easy lemma.

Lemma 4.9. *The natural functor $p\colon \tau_{\leq 0}\mathcal{B} \to H^0\mathcal{B}$ is a quasi-equivalence.*

Proof. The functor is the identity on the objects, so we only have to prove that it induces isomorphisms the homologies of the hom-spaces. By definitions those vanish positive degree, and p induces an isomomorphism in degree 0. Therefore, the claim will follow if we prove that $H^i\mathcal{B}(Y,Z) = 0$ for all $Y, Z \in \mathcal{B}$ and $i < 0$. Recalling that the objects of $\mathcal{B}$ are those of the form $\epsilon\pi(h_Z)$ for some $Z \in A$, we have

$$\begin{aligned} H^i\mathcal{B}(X,Y) =& H^i\mathcal{C}(\epsilon\pi(h_W), \epsilon\pi(h_Z)) = H^0\mathcal{C}(\epsilon\pi(h_W), \epsilon\pi(h_Z)[i]) \cong \\ &\cong \mathrm{Hom}_{D(A)/_L}(\pi(h_W), \pi(h_Z)[i]) = 0, \end{aligned}$$

by condition c). $\square$

An intermediate result

We now prove an intermediate result that will not be needed in the proof, but has some interest of its own sake.

Proposition 4.10. *Let $\mathcal{A}$ be a small k-linear category considered as a dg-category concentrated in degree 0. Then the dg-derived category $D(\mathcal{A})$ has a unique enhancement.*

Proof. Let $(\mathcal{C}, \epsilon)$ be an enhancement of $D(\mathcal{A})$, and denote as before with $\mathcal{B}$ its full dg-subcategory whose objects correspond to those of $\mathcal{A}$ via the functors

$$\mathcal{A} = H^0\mathcal{A} \to D(\mathcal{A}) \xrightarrow{\epsilon} H^0\mathcal{C}.$$

Since both functors are fully faithful, we have an equivalence $A \xrightarrow{\sim} H^0\mathcal{B}$. We have again a zig-zag of dg-functors

$$\mathcal{A} \cong H^0\mathcal{B} \xleftarrow{p} \tau_{\leq 0}\mathcal{B} \xrightarrow{i} \mathcal{B}.$$

This time however, it is easy to prove that $\mathcal{B}$ has homology concentrated in degree 0, and therefore both p and i are quasi-equivalences. Indeed, for any $X, Y \in \mathcal{B}$, we have

$$\begin{aligned} H^n\mathcal{B}(X,Y) \cong& \mathcal{C}(X, Y[n]) \cong D(\mathcal{A})(\epsilon X, \epsilon Y[n]) \cong D(A)(h_A, h_B[n]) \\ \cong& H^n\mathcal{A}(A,B) = 0 \end{aligned}$$

for some $A, B \in \mathcal{A}$. This immediately implies that p and i are quasi-equivalences, so we also have a zig-zag of quasi-equivalences

$$\mathcal{SF}(A) \xleftarrow{\mathrm{Ind}_p} \mathcal{SF}(\tau_{\leq 0}\mathcal{B}) \xrightarrow{\mathrm{Ind}_i} \mathcal{SF}(\mathcal{B}).$$

that defines an isomorphism in **Hqe**

$$\mathcal{SF}(\mathcal{A}) \xrightarrow{\sim} \mathcal{SF}(\mathcal{B}).$$

Since by construction the objects of $H^0\mathcal{B} \cong A$ form a compact set of generators for $H^0\mathcal{C} \cong D(\mathcal{A})$, lemma 4.8 gives that the natural quasi-functor $\mathcal{C} \to \mathcal{SF}(\mathcal{B})$ is an equivalence. We have then shown that any enhancement of $D(\mathcal{A})$ is equivalent to the canonical enhancement $\mathcal{SF}(\mathcal{A})$. This concludes the proof. □

As the above example shows, the dg-part of the proof is done: we have constructed a "good" quasi-functor

$$\rho\colon \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B}).$$

We still need to show that it induces an equivalence

$$\mathcal{SF}(\mathcal{A}) / \mathcal{SF}(\mathcal{A}) \cap \mathcal{L} \to \mathcal{SF}(\mathcal{B})$$

which, by Theorem 3.88, is equivalent to proving that

$$H^0\rho\colon D(\mathcal{A}) \to D(\mathcal{B})$$

induces an equivalence

$$D(\mathcal{A}) / L \to D(\mathcal{B}).$$

This is done in steps 3 and 4.

Proof of step 3

We have to prove that the quasi-functor

$$\rho\colon \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B})$$

factors through the quotient $\mathcal{SF}(\mathcal{A}) / \mathcal{L} \cap \mathcal{SF}(\mathcal{A})$, i.e. that the associated functor

$$F_1 = H^0\rho\colon D(\mathcal{A}) \to D(\mathcal{B}).$$

annihilates L. For this we will need several intermediate results; we begin with a construction. For a given complex C, we can define its stupid truncations as the complexes

$$\sigma_{\le m}C = \cdots \to C^{m-1} \to C^m \to 0 \to \cdots$$

and

$$\sigma_{\geq n} C = \cdots \to 0 \to C^n \to C^{n+1} \to \cdots$$

We also define

$$C^{[n,m]} = \sigma_{\geq n}\sigma_{\leq m} C = \sigma_{\leq m}\sigma_{\geq n} C = \cdots \to 0 \to C^n \to \cdots \to C^m \to 0 \to \cdots$$

For any fixed m it is easy to show that $\sigma_{\leq m} C \cong \varinjlim_n C^{[n,m]}$. We now want to apply these constructions to semi-free dg $\mathcal{A}$-modules. First, observe that since $\mathcal{A}$ is concentrated in degree 0, a representable $\mathcal{A}$-module h_A is also concentrated in degree 0, in the sense that $h_A(B)$ is concentrated in degree 0 for any $B \in \mathcal{A}$. Therefore, any free dg $\mathcal{A}$-module C is an actual complex

$$\cdots \to C^n \to C^{n+1} \to C^{n+2} \to \cdots$$

where every C^i is a direct sum of shifts by i of representables and the differential is zero. From this follows that any semi-free dg $\mathcal{A}$-module P is a chain complex

$$\cdots \to P^n \to P^{n+1} \to P^{n+2} \to \cdots$$

where each P^i is a direct sum of shifts by i of representables. This time however we have no information about the differential, since the sequences in the definition of semi-free module split only at the graded level. Furthermore, any bounded above complex of this form is automatically semi-free, by considering the filtration induced by the grading; we will denote with $\mathcal{SF}^-(\mathcal{A})$ the full dg-subcategory of $\mathcal{M}od\text{-}\mathcal{A}$ composed by bounded above semi-free dg $\mathcal{A}$-modules. Now, for any dg $\mathcal{A}$-module P we can define its the truncations $\sigma_{\leq m} P$ and $\sigma_{\geq n} P$ by via the obvious definitions on objects, and in the same way we can define $P^{[n,m]}$. The discussion above implies that the truncations of a semi-free $\mathcal{A}$-modules are again semi-free. Furthermore, we have an isomorphism $\sigma_{\leq m} P \cong \varinjlim_n P^{[n,m]}$ in $\mathcal{CA}$. If P is semi-free, this implies that there is also an isomorphism $\sigma_{\leq m} P \cong \underrightarrow{\mathrm{hocolim}}_n P^{[n,m]}$ in $H^0\mathcal{SF}(\mathcal{A}) \cong D(\mathcal{A})$.

Lemma 4.11. *Let $F \colon H^0\mathcal{SF}(\mathcal{A}) \to \mathcal{T}$ be an exact functor between triangulated categories, and suppose that F satisfies the following conditions:*

1. *F preserves direct sums.;*
2. *$F(h_A)$ is compact in $\mathcal{T}$ for any $A \in \mathcal{A}$;*
3. $\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(h_B)[i]) = 0$ *for any A, B in $\mathcal{A}$ and $i < 0$.*

Then, for any $A \in \mathcal{A}$ and $P \in \mathcal{SF}(\mathcal{A})$ we have

$$\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(\sigma_{\geq n} P)[i]) = 0$$

for any $i < n$.

Proof. The filtration F_j on P induces a filtration $E_j = F_j \cap \sigma_{\leq m} P$ on $\sigma_{\leq m} P$, in a way that the quotient E_{j+1}/E_j has the form $\bigoplus_k h_{B_k}[s_k]$ for some B_k in $\mathcal{A}$ and $s_k \leq -n$. We prove by induction that

$$\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(E_{j+1}/E_j)[i]) = 0$$

for each j. The base step is obvious since $E_0 = 0$. For the inductive step, recall that since F is exact we have a a distinguished triangle

$$F(E_i)[i] \to F(E_{i+1})[i] \to F(E_{i+1}/E_i)[i] \to F(E_i)[i+1],$$

so it is sufficient to prove that

$$\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(E_{i+1}/E_i)[i]) = 0$$

for any i. Using hypotheses 1-3, we have

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(E_{j+1}/E_j)[i]) = \mathrm{Hom}_{\mathcal{T}}(F(h_A), F(\bigoplus_k h_{B_k}[s_k])[i]) \cong \\ &\cong \mathrm{Hom}_{\mathcal{T}}(F(h_A), \bigoplus_k F(h_{B_k}[s_k+i])) \cong \bigoplus_k \mathrm{Hom}_{\mathcal{T}}(F(h_A), F(h_{B_k}[s_k+i])) \cong \\ &\cong \bigoplus_k \mathrm{Hom}_{\mathcal{T}}(F(h_A), F(h_{B_k})[s_k+i]) = 0, \end{aligned}$$

since $s_k + i < n$. To conclude, recall that $P \cong \varinjlim_j E_j$ in $Z^0\mathcal{SF}(\mathcal{A})$, so $P \cong \underrightarrow{\mathrm{hocolim}}\, E_j$ in $H^0\mathcal{SF}(\mathcal{A})$. Since F is exact and preserves coproducts, it also preserves homotopy colimits. Therefore, $F(P) \cong \underrightarrow{\mathrm{hocolim}}_j\, F(E_j)$. Now, a direct application of Lemma 1.22 finishes the proof. □

Corollary 4.12. *In the same hypotheses of Lemma 4.11, for every $A \in \mathcal{A}$, $P \in \mathcal{SF}(\mathcal{A})$ and every $m \geq 0$ we have an injection*

$$\mathrm{Hom}_{\mathcal{T}}(F(h_A), F(P)) \hookrightarrow \mathrm{Hom}_{\mathcal{T}}(F(h_A), F(\sigma_{\leq m} P))$$

If $m > 0$, this is a bijection.

Proof. This follows immediately from what we just proved and from the existence of the distinguished triangle in $H^0\mathcal{SF}(\mathcal{A})$

$$\sigma_{\geq (m+1)} P \to P \to \sigma_{\leq m} P \to \sigma_{\geq (m+1)} P[1].$$

□

Proposition 4.13. *Let $F_1, F_2 \colon H^0\mathcal{SF}^-(\mathcal{A}) \to \mathcal{T}$ be two exact functors satisfying the hypotheses of Lemma 4.11. Suppose also that there is a natural isomorphism $F_1 \circ h_{\mathcal{A}} \cong F_2 \circ h_{\mathcal{A}}$. Then for every $P \in \mathcal{SF}^-(\mathcal{A})$ there exists an isomorphism*

$$\theta_P \colon F_1(P) \xrightarrow{\sim} F_2(P)$$

such that for any $A \in \mathcal{A}$ and every $k \in \mathbb{Z}$ and $f \in \operatorname{Hom}_{H^0\mathcal{SF}^-(\mathcal{A})}(h_A[-k], P)$ with $k \in \mathbb{Z}$ the diagram

$$\begin{array}{ccc} F_1(h_A)[-k] & \xrightarrow{F_1(f)} & F_1(P) \\ {\scriptstyle \theta_A[-k]}\downarrow & & \downarrow{\scriptstyle \theta_P} \\ F_2(h_A)[-k] & \xrightarrow{F_1(f)} & F_2(P) \end{array} \tag{4.2}$$

commutes.

We omit the full proof of this Proposition, referring to [LO10, Proposition 3.4] for the details. We still give below a brief sketch of the construction of the isomorphism θ_P.

Sketch of the proof. The isomorphism θ_P is constructed in steps: first one uses the fact that F_1 and F_2 preserve coproducts to extend θ from the representables (where it exists by hypothesis) to the full subcategory formed by coproducts of representables. Then one considers an arbitrary bounded above semi-free dg $\mathcal{A}$-module

$$\cdots \to P^{m-1} \to P^m \to 0,$$

and uses the existence of the distinguished triangle

$$P^{n-1}[-n] \to \sigma_{\geq n}P \to \sigma_{\geq n+1} \to P^{n-1}[-n+1]$$

to define, by descending induction on $n \leq m$, an isomorphism

$$\theta_{\geq n} \colon F_1(\sigma_{\geq n}P) \xrightarrow{\sim} F_2(\sigma_{\geq n}P).$$

At this point, one uses the fact that $P \cong \underrightarrow{\operatorname{hocolim}}_n \sigma_{\geq n}P$ to extend the isomorphism to all of $\mathcal{SF}^-(\mathcal{A})$. Finally, one proves that the just defined isomorphism makes diagram (4.2) commute. □

Recall now that we had defined the dg functor

$$a \colon \mathcal{A} \to H^0\mathcal{B}$$

that, by a slight abuse of notation (since the image of ϵ is not in general contained in $H^0\mathcal{B}$), can be represented as the composition

$$\mathcal{A} = H^0\mathcal{A} \to D(\mathcal{A}) \to {}^{D(\mathcal{A})}\!/_{L} \xrightarrow{\epsilon} H^0\mathcal{B}$$

Recall also that the quasi-functor

$$\rho\colon \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B})$$

was defined as the zig-zag

$$\mathcal{SF}(A) \overset{\mathrm{Ind}_a}{\to} \mathcal{SF}(H^0\mathcal{B}) \overset{\mathrm{Ind}_p}{\leftarrow} \mathcal{SF}(\tau_{\leq 0})\mathcal{B} \overset{\mathrm{Ind}_i}{\to} \mathcal{SF}(\mathcal{B}).$$

We call $F_1 = H^0\rho\colon D(\mathcal{A}) \to D(\mathcal{B})$. It is easy to prove that F_1 satisfies conditions 1 to 3 of Lemma 4.11: for condition 1, observe that Ind_p is a quasi-equivalence, while Ind_a and Ind_i are right adjoints and thus preserve coproducts. Condition 2 follows from the fact that F_1 sends $H^0\mathcal{A} \subseteq D(\mathcal{A})$ to $H^0\mathcal{B} \subseteq D(\mathcal{B})$, and the objects of $H^0\mathcal{B}$ are always compact in $D(\mathcal{B})$. Condition 3 follows from the equivalent hypothesis imposed on the quotient $D(\mathcal{A}) \overset{\pi}{\to} {}^{D(\mathcal{A})}\!/_{L}$. Define now the functor F_2 as the composition

$$D(\mathcal{A}) \overset{\pi}{\to} {}^{D(\mathcal{A})}\!/_{L} \overset{\epsilon}{\to} H^0\mathcal{C} \overset{H^0\varphi}{\to} D(\mathcal{B}).$$

F_2 also satisfies conditions 1-3: by Lemma 4.7 it preserves coproducts; Furthermore, by definition of a the diagram

$$\begin{array}{ccccccccc} & & & & \overset{F_2}{\frown\!\!\!\longrightarrow} & & & & \\ \mathcal{A} & \hookrightarrow & D(\mathcal{A}) & \longrightarrow & {}^{D(\mathcal{A})}\!/_{L} & \xrightarrow{\sim} & H^0\mathcal{C} & \xrightarrow{H^0\varphi} & D(\mathcal{B}) \\ & \searrow^{a} & & & & & & \nearrow & \\ & & & & H^0\mathcal{B} & \hookrightarrow & & & \end{array} \tag{4.3}$$

is commmutative, so F_2 sends objects of $\mathcal{A}$ to images via the derived Yoneda embedding of objects of $\mathcal{B}$, which are always compact. Finally, condition 3 follows from the equivalent condition imposed to the quotient.

It is easy to prove the following:

Lemma 4.14. *There exists a natural isomorphism*

$$\theta\colon F_1 \circ h_{\mathcal{A}} \overset{\sim}{\to} F_2 \circ h_{\mathcal{A}}.$$

Proof. The commutativity diagram (4.3) is equivalent to saying the the composition

$$\mathcal{A} \hookrightarrow D(\mathcal{A}) \xrightarrow{F_2} D(\mathcal{B})$$

coincides with

$$\mathcal{A} \xrightarrow{a} H^0\mathcal{B} \hookrightarrow D(\mathcal{B}).$$

Recall that ρ (and therefore F_1) was constructed starting from the zig-zag

$$\mathcal{A} \xrightarrow{\ a\ } H^0\mathcal{B} \xleftarrow{\ p\ } \tau_{\leq 0}\mathcal{B} \xrightarrow{\ i\ } \mathcal{B}.$$

By Proposition 3.37, the induced diagram

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{\quad a \quad} & H^0\mathcal{B} & \xleftarrow{\quad p \quad} & \tau_{\leq 0}\mathcal{B} & \xrightarrow{\quad i \quad} & \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{SF}(\mathcal{A}) & \xrightarrow[\operatorname{Ind}_a]{} & \mathcal{SF}(H^0\mathcal{B}) & \xleftarrow[\operatorname{Ind}_p]{} & \mathcal{SF}(\tau_{\leq 0}\mathcal{B}) & \xrightarrow[\operatorname{Ind}_i]{} & \mathcal{SF}(\mathcal{B})
\end{array}$$

commutes up to isomorphism. Passing to the homotopy categories and using the fact that $H^0\tau_{\leq 0}\mathcal{B} = H^0(H^0\mathcal{B}) = H^0\mathcal{B}$, we get that the diagram

$$\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{\quad a \quad} & H^0\mathcal{B} & = & H^0\mathcal{B} & = & H^0\mathcal{B} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D(\mathcal{A}) & \longrightarrow & D(H^0\mathcal{B}) & \longleftarrow & D(H^0\mathcal{B}) & \longrightarrow & D(\mathcal{B})
\end{array}$$

commutes (always up to isomorphism). The commutativity of the outer rectangle, together with the observation made at the beginning of the proof, concludes the argument. ∎

We then have the following proposition, that concludes the proof of step 3.

Proposition 4.15. *$F_1\colon D(\mathcal{A}) \to D(\mathcal{B})$ annihilates $L \subseteq D(\mathcal{A})$. Therefore, the quasi-functor ρ factors through the quotient* $\mathcal{SF}(\mathcal{A})/_{\mathcal{L} \cap \mathcal{SF}(\mathcal{A})}$.

Proof. In this proof the crucial observation is that, by definition, the functor F_2 annihilates L. So what we have to do is use this information to prove that F_1 does as well. Since F_1 sends surjectively $H^0\mathcal{A} \subseteq D(\mathcal{A})$ to $H^0\mathcal{B} \subseteq D(\mathcal{B})$, the set $\{F_1(h_A)\}_{A \in \mathcal{A}}$ is a set of compact generators for $D(\mathcal{B})$. Take now any $P \in \mathcal{SF}(\mathcal{A}) \cap \mathcal{L}$. To prove that $F_1(P) = 0$ it is therefore sufficient to prove that

$$\operatorname{Hom}_{D(\mathcal{B})}(F_1(h_A)[k], F_1(P)) = 0$$

for any $k \in \mathbb{Z}$ and $A \in \mathcal{A}$. Consider the truncation $\sigma_{\leq m} P$ for some $m \geq k$. By Corollary 4.12, we have a bijection

$$\mathrm{Hom}_{D(\mathcal{B})}(F_1(h_A)[k], F_1(P)) \cong \mathrm{Hom}_{D(\mathcal{B})}(F_1(h_A)[k], F_1(\sigma_{\leq m} P));$$

this is useful since $\sigma_{\leq m} P \in \mathcal{SF}^-(\mathcal{A})$, and we can apply Proposition 4.13 to find an isomorphism $F_1(\sigma_{\leq m} P) \cong F_2(\sigma_{\leq m} P)$. Of course we also have an isomorphism $F_1(h_A) \cong F_2(h_A)$. Therefore,

$$\mathrm{Hom}_{D(\mathcal{B})}(F_1(h_A)[k], F_1(P)) \cong \mathrm{Hom}_{D(\mathcal{B})}(F_2(h_A)[k], F_2(\sigma_{\leq m} P))$$

and again by Corollary 4.12

$$\mathrm{Hom}_{D(\mathcal{B})}(F_2(h_A)[k], F_2(\sigma_{\leq m} P)) \cong \mathrm{Hom}_{D(\mathcal{B})}(F_2(h_A)[k], F_2(P)) \cong 0.$$

This concludes the proof. □

Proof of step 4

We have proved that the quasi-functor ρ factors through a quasi-functor

$$\tilde{\rho} \colon \mathcal{SF}(\mathcal{A}) / \mathcal{L} \cap \mathcal{SF}(\mathcal{A}) \to \mathcal{SF}(\mathcal{B}).$$

Call F the exact functor

$$H^0 \tilde{\rho} \colon D(\mathcal{A}) / L \to D(\mathcal{B}).$$

We already know that the functor F_1 is isomorphic to the composition $F \circ \pi$. In order to prove that $\tilde{\rho}$ is an isomorphism (and hence the theorem) it only remains the following

Proposition 4.16. *The functor* $F \colon D(\mathcal{A}) / L \to D(\mathcal{B})$ *is an equivalence.*

Proof. Be begin with full faithfulness. By Lemma 1.33, The elements $\{\pi h_B\}_{B \in \mathcal{B}}$ form a set of compact generators for $D(\mathcal{A}) / L$. Moreover, the functor F coincides with F_1 on objects; therefore Lemma 4.14 and diagram (4.3) show that F_1 carries (images via the quotient of) representable $\mathcal{A}$-modules to representable $\mathcal{B}$-modules.

So, by Lemma 1.19, to prove that F is fully faithful it is sufficient to prove that

$$F \colon \mathrm{Hom}_{D(\mathcal{A}) / L}(\pi h_A, \pi h_B) \to \mathrm{Hom}_{D(\mathcal{B})}(F \pi h_A, F \pi h_B)$$

is a bijection for all $A, B \in \mathcal{A}$. We proceed by several simplifications. First, we recall that π admits a fully faithful right adjoint μ. Since μ is fully faithful, the counit

$$\mathrm{id}_{D(\mathcal{A})/L} \to \pi\mu$$

is an isomorphism. Call $P = \mu\pi h_B \in D(\mathcal{A})$; we can clearly suppose P to be semi-free. We then have a natural isomorphism $h_B \to P$, so $\pi h_B \cong \pi P$. Moreover, $P \in L^{\perp}$, since for any $K \in L$ we have

$$\mathrm{Hom}_{D(\mathcal{A})}(K, P) = \mathrm{Hom}_{D(\mathcal{A})}(K, \mu\pi h_B) \cong \mathrm{Hom}_{D(\mathcal{A})/L}(\pi K, \pi h_B) = 0.$$

Therefore by Proposition 1.31,

$$\pi\colon \mathrm{Hom}_{D(\mathcal{A})}(h_A, P) \to \mathrm{Hom}_{D(\mathcal{A})/L}(\pi h_A, \pi P)$$

is an isomorphism. Consider now the stupid truncation $\sigma_{\leq m} P$, for some $m > 0$. We then have a short exact sequence in $Z^0\mathcal{SF}(\mathcal{A})$

$$0 \to \sigma_{\geq m+1} P \to P \to \sigma{\leq_m} P$$

that gives an exact triangle

$$\sigma_{\geq m+1} P \to P \to \sigma_{\leq m} P \to \sigma_{\geq m+1} P[1]$$

in $D(\mathcal{A})$. Since by the dg-Yoneda lemma

$$\mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\geq m+1} P) \cong H^0 \sigma_{\geq m+1} P(A) = 0,$$

we get that the natural map

$$\mathrm{Hom}_{D(\mathcal{A})}(h_A, P) \to \mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m} P)$$

is an isomorphism. Consider then the commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{D(\mathcal{A})}(h_A, P) & \xrightarrow{\sim} & \mathrm{Hom}_{D(\mathcal{A})/L}(\pi h_A, \pi P) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m} P) & \xrightarrow[\pi]{} & \mathrm{Hom}_{D(\mathcal{A})/L}(\pi h_A, \pi\sigma_{\leq m} P) \end{array}$$

The right vertical arrow is an isomorphism by Corollary 4.12, so the lower horizontal arrow is as well. Recall now that by Lemma 4.14 and Proposition 4.13 there are isomorphisms

$$\theta_A\colon F_1(h_A) \to F_2(h_A) \quad \text{and} \quad \theta_{\leq m}\colon F_1(\sigma_{\leq m} P) \to F_1(\sigma_{\leq m} P)$$

such that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m}P) & \xlongequal{\qquad} & \mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m}P) \\
{\scriptstyle F_1}\downarrow & & \downarrow{\scriptstyle F_2} \\
\mathrm{Hom}_{D(\mathcal{B})}(F_1(h_A), F_1(\sigma_{\leq m}P)) & \xrightarrow{\theta_{\leq m}\circ\theta_A^{-1}} & \mathrm{Hom}_{D(\mathcal{B})}(F_2(h_A), F_2(\sigma_{\leq m}P))
\end{array}$$

commutes. Since $F_1 = F \circ \pi$ and $F_2 = H^0(\varphi) \circ \epsilon \circ \pi$ we get the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m}P) & \xlongequal{\qquad} & \mathrm{Hom}_{D(\mathcal{A})}(h_A, \sigma_{\leq m}P) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\mathrm{Hom}_{D(\mathcal{A})/L}(\pi h_A, \pi\sigma_{\leq m}P) & \xlongequal{\qquad} & \mathrm{Hom}_{D(\mathcal{A})/L}(\pi h_A, \pi\sigma_{\leq m}P) \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle H^0\varphi\circ\epsilon} \\
\mathrm{Hom}_{D(\mathcal{B})}(F_1(h_A), F_1(\sigma_{\leq m}P)) & \xrightarrow{\theta_{\leq m}\circ\theta_A^{-1}} & \mathrm{Hom}_{D(\mathcal{B})}(F_2(h_A), F_2(\sigma_{\leq m}P))
\end{array}$$

The key observation now is that all the arrows, except possibly F, are isomorphisms: we have already seen that π is an isomorphism, and by Lemma 4.8 $H^0\varphi$ is fully faithful. Therefore F as well is an isomorphism, and we have proved full faithfulness. Essential surjectivity is proved as usual: first we observe that, since F_1 preserves coproducts and F coincides with F_1 on objects, F preserves coproducts as well. Therefore the image of F is a localizing subcategory of $D(\mathcal{B})$ that contains $H^0\mathcal{B}$, so it has to be all of $D(\mathcal{B})$. □

4.4 Further results

In the years following the publication of [LO10], several generalizations have been found. In this section, we give a short overview of those.

An important observation is that the proofs of [LO10] rely heavily on the compact generation of some triangulated categories. The problem is that this class is not very flexible: in general, quotients of compactly generated categories might fail to be compactly generated. Moreover, it was known (see [Nee01a]) that derived categories of sheaves could fail to be compactly generated. On the other hand, Neeman had introduced in [Nee01b] the notion of a well-generated triangulated category, which can be considered as a generalization of that of compactly generated category to uncountable cardinals. Well generated triangulated categories have several desirable properties: to begin with, one can prove that the derived category of any Grothendieck category is always well generated, and that well generated triangulated categories are closed under localizations; crucially, well generated triangulated categories also admit a form of Brown's representability theorem. Applying these techniques, Canonaco and Stellari in [CS18] proved the following generalization of Corollary 4.5:

Theorem 4.17. *The derived category $D(\mathcal{C})$ of any Grothendieck abelian category admits a unique dg-enhancement.*

This has the very pleasing consequence

Corollary 4.18. *Let X be any scheme. Then the derived category $D(\operatorname{Qcoh} X)$ has a unique enhancement.*

The strategy of the proof of Theorem 4.17 is very similar to that of Corollary 4.5: it relies on a technical result analogous to Theorem 4.2, whose proof has the same structure to the one we saw; the main differences comes from the fact that, in [CS18], the authors work with well generated triangulated categories rather than with compactly generated ones. This causes some noticeable differences in the proof. Note also that here (as well as in all the subsequent papers) k is allowed to be an arbitrary commutative ring, and not necessarily a field.

Stable ∞-enhancements

We have already recorded the fact that there exist several different types of enhancements besides dg-enhancements; one that has been gaining popularity is Lurie's theory of stable ∞-categories. We do not give a full definition of those; for our very limited scopes, it is sufficient to say that any ∞-category $\mathcal{C}$ induces an ordinary category $\operatorname{Ho}\mathcal{C}$, called the homotopy category of $\mathcal{C}$; if $\mathcal{C}$ is stable[1], then $\operatorname{Ho}\mathcal{C}$ is in a natural way triangulated (so in particular additive). One can then can talk about stable ∞-enhancement of triangulated categories: as in the case of dg-categories (in fact, arguably even more naturally) there is a notion of equivalence of ∞-categories; therefore, it makes sense to talk about uniqueness of stable ∞-enhancements.

[1] Note that being stable is a property, not extra structure.

In a very imprecise sense, stable ∞-categories can be considered as generalizations of dg-categories to non-linear settings; it has been proven (see for example [Coh16]) that, if we impose to stable ∞-categories the extra structure of k-linearity for a field k (namely, by enriching them in Hk-module symmetric spectra), they become equivalent - in a suitable sense - to dg-categories. However, even if the theory is in principle equivalent, the practice (and language) of ∞-categories and dg-categories is very different. In

[Ant18], Antieau proved the uniqueness of the stable ∞-enhancement of several triangulated categories of algebraic origin; furthermore, he proved that this implies that they also have a unique dg-enhancement. To make an example, he proved the following:

Theorem 4.19. *Let $\mathcal{C}$ be a small abelian category. Then $D^b(\mathcal{C})$ admits a unique stable ∞-enhancement.*

The reader will notice that this is the fist time that we talk about enhancements of bounded derived categories; this is not because their study is not present in the literature; in both [LO10] and [CS18] appear results about the uniqueness of enhancements of bounded derived categories. However, the techniques shown in this thesis do not make it particularly easy to study those[2]. As a consequence, the results prior to Theorem 4.19 had various technical hypotheses attached to them, mainly in order to trace back to the more manageable case of $\left(D(\mathcal{A})/L\right)^c$, the subcategory of compact objects of a quotient of the dg-derived category of an ordinary category.

[2]Possibly for the fact that the derived category of a dg-category is by definition unbounded.

The proofs in [Ant18] are very different in spirit to those in [LO10] and [CS18]; they rely on the theory of prestable ∞-categories developed by Lurie in [Lur18, Appendix C]. In particular, they make heavy use of t-structures in triangulated categories (see [GM02]) and of a Gabriel-Popescu theorem for prestable ∞-categories. In [GG21], the authors prove similar results for dg-categories, and apply them to recover the uniqueness of the enhancement for the derived category of a Grothendieck category.

Back to the dg-world

Very recently, in [CNS21] Canonaco, Neeman and Stellari, going back to the dg-setting, have shown several generalizations of the theorems above. In particular, they proven the following

Theorem 4.20. *Let $\mathcal{C}$ be a small abelian category. Then the derived categories $D(\mathcal{C})$, $D^b(\mathcal{C})$, $D^+(\mathcal{C})$ and $D^-(\mathcal{C})$ all have unique dg-enhancements.*

Since for any small abelian category $\mathcal{C}$ there exists a small abelian category $\mathcal{A}$ with an equivalence $D(\mathcal{A}) \cong \mathcal{K}(\mathcal{C})$ (see [CNS21, Remark 1.3]), Theorem 4.20 implies the following

Corollary 4.21. *Let $\mathcal{C}$ be an abelian category. Then the homotopy categories $\mathcal{K}(\mathcal{C})$, $\mathcal{K}^b(\mathcal{C})$, $\mathcal{K}^+(\mathcal{C})$ and $\mathcal{K}^-(\mathcal{C})$ all have unique dg-enhancements.*

Their paper contains both new proofs of theorems present in [Ant18] and new results; in particular, the uniqueness of the enhancement for $D(\mathcal{C})$ for an arbitrary abelian category is a new result. All the proofs in [CNS21] are done entirely in the realm of dg-categories, using ideas and techniques recalling those shown in this thesis. Crucially though, the authors use a completely different type of generation that allows them to discard all the hypotheses on $\mathcal{C}$, but makes the proof considerably more technically demanding.

Now what?

In a sense, Theorem 4.20 closes a decade-old question; at this point there are no obvious generalization to be proven; it is known that quotients and subcategories of categories with a unique enhancement might fail to have a unique enhancement, so there is no hope for any far reaching generalization in that direction. There are, of course, still a lot of open questions; for example, in [CNS21] the authors make several examples of categories for which it is not know whether they admit a unique enhancement, namely categories of matrix factorizations linear over a field and admissible[3] subcategories of categories with a unique enhancement that are again linear over a field; in both cases, if one drops the hypothesis for k to be a field, counterexamples are known.

[3]A triangulated subcategory admissible if the inclusion functor admits both a left and a right adjoint.

Exact functors and lifts

A very natural question that we have not approached is that of the existence of lifts of exact functors: given an exact functor between the homotopy categories of two pretriangulated dg-categories, does it lift to a quasi-functor between the two dg-categories? Here, the situation is less ideal than in the case of enhancements. While the space of quasi-functors between dg-categories has a very explicit description (see Theorem 3.83), exact functors between triangulated categories are harder to characterize: most functors that one encounters in practice admit lifts, but even in very natural cases have been found examples (see [RBN19] and [Nee92a]) of exact functors that do not admit a lift. There exist some partial results of existence of lifts (mainly in the case of fully faithful functors) but no general characterization.

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Symbols

$\mathrm{Hom}_{\mathcal{C}}(A,B)$ or $\mathcal{C}(A,B)$	Space of morphisms between objects A and B in a category $\mathcal{C}$
$\mathrm{Hom}_k(A,B)$	Set of morphisms between the k-modules A and B
$\mathcal{C}[\mathcal{S}^{-1}]$	Localization of a category $\mathcal{C}$ at a class of morphisms $\mathcal{S}$
Z^nA	n-cycles of a chain complex A
B^nA	n-boundaries of a chain complex A
H^nA	n-th homology of a chain complex A
$\mathbf{C}(k)$	Category of chain complexes of k-modules A
$\mathcal{K}(k)$	Homotopy category of chain complexes of k-modules
$\mathbf{C}(A)$	Category of chain complexes in an abelian category A
$\mathcal{K}(A)^+$, $\mathcal{K}(A)^-$, $\mathcal{K}(A)^b$ and $\mathcal{K}(A)$	Bounded below, bounded above, bounded and unbounded homotopy category of an abelian category A
$D(A)^+$, $D(A)^-$, $D(A)^b$ and $D(A)$	Bounded below, bounded above, bounded and unbounded derived category of an abelian category A
$D(\mathcal{A})$	Derived category of a dg-category $\mathcal{A}$
$\bullet \to \bullet$	Walking arrow category
$\mathrm{Mor}(\mathcal{C})$	Category of morphism of a category $\mathcal{C}$
Ab	Category of abelian groups
$\underrightarrow{\mathrm{hocolim}}\, X_i$	Homotopy colimit of a sequence X_i
$\mathcal{F}_{\mathcal{T}}\mathcal{S}$ or $\mathcal{F}\mathcal{S}$	Class of morphisms in $\mathcal{T}$ whose cone lies in a triangulated subcategory $\mathcal{S}$
${}^{\mathcal{A}}\!/_{\mathcal{B}}$	Verdier quotient of a triangulated category $\mathcal{A}$ by a triangulated subcategory $\mathcal{B}$, or Drinfeld quotient of a dg-category $\mathcal{A}$ by a full dg-subcategory $\mathcal{B}$

$\mathcal{S}^{\perp}$ and ${}^{\perp}\mathcal{S}$	Orthogonal subcategories to a triangulated subcategory $\mathcal{S}$
$B \wedge I$	Cylinder object for an object B
B^I	Path object for an object B
$\mathcal{C}_{cf}$	Full subcategory of bifibrant objects of a model category $\mathcal{C}$
$\mathbf{Vect}_{\mathbb{K}}$	Category of vector spaces over a field $\mathbb{K}$
Set	Category of sets
k-**Mod**	Category of modules over a commutative ring k
$\mathcal{H}om(A, B)$	Internal hom of chain complexes, or dg-categories of dg-functors between dg-categories
$[A, B]$	Internal hom in an arbitrary closed monoidal category
$\otimes$	Tensor product of chain complexes, k-modules of dg-categories
$\mathcal{A}^0$	Category with the same objects of the dg-category $\mathcal{A}$ and with morphism being the degree 0 morphisms of $\mathcal{A}$
$Z^0\mathcal{A}$ and $Z^0\mathcal{F}$	Underlying category of the dg-category $\mathcal{A}$, or functor between the underlying categories by a dg-functor $\mathcal{F}$
$H^0\mathcal{A}$ and $H^0\mathcal{F}$	Homotopy category of the dg-category $\mathcal{A}$, or functor between the homotopy categories induced by a dg-functor $\mathcal{F}$
$\mathbf{dgcat}_k$	Category of small k-linear dg-categories, for a commutative ring k
$\mathcal{N}at_{\mathrm{dg}}(\mathcal{F}, \mathcal{G})$	Chain complex of graded natural transformation between the dg-functors $\mathcal{F}$ and $\mathcal{G}$
$\mathbf{C}_{\mathrm{dg}}(k)$	$\mathbf{C}(k)$ considered as a dg-category
$\mathbf{C}_{\mathrm{dg}}(A)$	$\mathbf{C}(A)$ considered as a dg-category
$\mathcal{A}^{op}$	Opposite dg-category of a dg-category $\mathcal{A}$
$\mathcal{M}od\text{-}\mathcal{A}$	Dg-category of right dg $\mathcal{A}$-modules
$\mathcal{H}\mathcal{A}$	$H^0\mathcal{M}od\text{-}\mathcal{A}$
$\mathcal{C}(\mathcal{A})$	$Z^0\mathcal{M}od\text{-}\mathcal{A}$
$\mathcal{M}od\text{-}\mathcal{A}_{gr}$	Abelian category of graded $\mathcal{A}$-modules
h_A and $\tilde{h}_A$	$\mathcal{A}(-, A)$ and $\mathcal{A}(A, -)$
$h_{\mathcal{A}}$	Dg-Yoneda embedding
$M \otimes_{\mathcal{A}} N$	Tensor product of a right dg $\mathcal{A}$-module and a left dg $\mathcal{A}$-module

$\mathcal{H}om(X,-)$	Dg-functor sending the right dg $\mathcal{B}$-module N to the right dg $\mathcal{A}$-module $\mathcal{N}at_{\mathrm{dg}}(X(A,-),N)$ for an $\mathcal{A}$-$\mathcal{B}$ bimodule X
$\mathrm{Res}_{\mathcal{F}}$	Restriction dg-functor
$\mathrm{Ind}_{\mathcal{F}}$	Induction dg-functor
$C(f)$	Several instances of a cone: the (strict) cone in the sense of homological algebra if f is a chain map or a dg-natural transformation, an arbitrary cone in the sense of triangulated categories otherwise
$\mathcal{A}^{pre\text{-}tr}$	Pretriangulated hull of a dg-category $\mathcal{A}$
$\mathcal{A}^{tr}$	$H^0\mathcal{A}^{pre\text{-}tr}$
$\overline{\mathcal{B}}$	Essential image of the dg-Yoneda embedding $\mathcal{B} \to \mathcal{M}od\text{-}\mathcal{B}$
$\underline{\mathcal{B}}$	Essential image of the derived Yoneda embedding $\mathcal{B} \to D(\mathcal{B})$
$\hat{\mathcal{B}}$	Essential image of the derived Yoneda embedding $\mathcal{B} \to \mathcal{H}\mathcal{B}$
$\mathrm{rep}(\mathcal{A},\mathcal{B})$	Full subcategory of $D(\mathcal{A}^{op}\otimes\mathcal{B})$ spanned by right quasi-representable bimodules
$\mathrm{rep}_{\mathrm{dg}}(\mathcal{A},\mathcal{B})$	Dg-enhancement of $\mathrm{rep}(\mathcal{A},\mathcal{B})$
$\mathrm{Tot}\, P_\bullet$	Total dg-module
h-proj($\mathcal{A}$)	Dg-category of h-projective right dg $\mathcal{A}$-modules
$\mathcal{SF}(\mathcal{A})$	Dg-category of semi-free right dg $\mathcal{A}$-modules
Hqe	Localization of $\mathbf{dgcat}_k$ at the quasi-equivalences
$\otimes^L$	Derived tensor product of dg-categories, or derived tensor product of dg $\mathcal{A}$-modules
$\mathcal{RH}om(\mathcal{A},\mathcal{B})$	Internal hom of $(\mathbf{Hqe},\otimes^L)$
$\mathbb{F}_p$	Field with p elements for a prime p
$\mathbf{T}_p$	Category of $\mathbb{F}_p$-modules considered as a triangulated category
$\mathbf{C}^{ai}_{\mathrm{dg}}(R\text{-Mod})$	Dg-category of acyclic complexes of R-modules with injective components
$\mathrm{Mod}\text{-}\mathcal{A}$	$\mathrm{Fun}_k(\mathcal{A},k\text{-Mod})$ for a k-linear category $\mathcal{A}$
$\sigma_{\leq m}A$ and $\sigma_{\geq m}A$	Stupid truncations of a chain complex or dg-module A
$A^{[n,m]}$	$\sigma_{\geq n}\sigma_{\leq m}A$

$\tau_{\geq 0}\mathcal{A}$ and $\tau_{\leq 0}\mathcal{A}$	Truncations of a chain complex or of a dg-category $\mathcal{A}$

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