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The curvature problem and deformations of triangulated categories

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[...] If I could only understand the beautiful consequences one can draw from the identity $d^2 = 0$.

J. Gwyn Griffiths, about the work of Henri Cartan

Contents

Chapter 1

Introduction

This thesis concerns the deformation theory of triangulated categories; these – often in their enhanced versions – are central objects in modern mathematics, playing a crucial role in fields such as derived algebraic geometry and homological mirror symmetry. While triangulated categories first emerged as a tool in algebraic geometry and in homotopy theory, they have become a topic of interest in and of themselves. The perspective we will be interested in is that of noncommutative algebraic geometry. Here, triangulated categories (or a subclass of those) act as a model for noncommutative schemes, where a scheme X is seen as a noncommutative scheme via its triangulated derived category $D(X)$. A surprising fact is that no satisfactory deformation theory for these objects exists. While there exist notions of deformation of a triangulated category, they tend to be in general not well-behaved. The issue is known in deformation theory as the *curvature problem*. This corresponds to the fact that roughly speaking, the Hochschild complex of a dg algebra (and, by extension, a triangulated category) encodes more deformations than the ones that one would a priori expect.

For the non-expert

In this short section I would like to motivate and briefly explain the content of this thesis to the reader who is not already a researcher in deformation theory or in a neighboring area[1]; while many of the results contained here have a technical flavour, the motivation does not.

Categories – of which triangulated categories are a specifically important example – are algebraic objects which have found wide applications in many fields of mathematics. Of particular interest for us is the fact that triangulated categories can be seen as a model for certain noncommutative spaces. Whatever precise meaning one might assign to this statement, the point is that one can also see a triangulated category as a geometric object – rather than just an algebraic one. A common theme in geometry is the importance of considering objects in *families*. It hence becomes a meaningful question to understand what a family (or a moduli space) of triangulated categories is. At the infinitesimal level, the question is hence to understand *deformations* of triangulated categories. This is not a new question, and deformations of triangulated categories have been studied by several

[1] But is still a mathematician; sorry, dad.

authors (e.g. Lurie [Lur11], Genovese-Lowen-Symons-Van den Bergh [GLSVdB24], Blanc-Katzarkov-Pandit [BKP18],...). However, the notion of deformation used there is not well-behaved. Very roughly, by the general theory one would expect a certain vector space – namely, the second Hochschild cohomology – to parametrize local (first order) deformations. The issue is that, in general, there are not enough deformations to realize the whole cohomology group. The defect is given by the so-called curved deformations, which are objects that are very hard to study via classical homological methods; for this reason, this is known as the *curvature problem.* What we propose in this thesis is a new kind of (first order) deformation theory for triangulated categories; the explicit aim is to realize the full Hochschild cohomology via infinitesimal deformations.

Consider the simple case of an algebra A over a field k; set $k[\varepsilon] = k[t]/(t^2)$. At a basic level, a first order deformation A_ε of A is given by deforming the multiplication of A into the star product

$$a \star b = ab + \varepsilon\mu(a, b)$$

defined on the $k[\varepsilon]$-module $A \oplus \varepsilon A$.

There are two ways to encode this. The first one is to say that A_ε is a ($k[\varepsilon]$-flat) $k[\varepsilon]$-algebra with the property that the reduction

$$A_\varepsilon \otimes_{k[\varepsilon]} k \cong A_\varepsilon/tA_\varepsilon$$

is isomorphic to A. Equivalently, one could impose the existence of the short exact sequence (rather, square-zero extension)

$$0 \to A \to A_\varepsilon \to A \to 0.$$

For algebras, these two conditions are easily seen to be equivalent. The classical definition of deformation for triangulated categories is a more or less straightforward extension of the first condition: one defines a (classical) deformation of a k-linear triangulated category $\mathcal{C}$ to be some $k[\varepsilon]$-linear category $\mathcal{C}_\varepsilon$ which reduces appropriately to $\mathcal{C}$. In Chapter 4 we define a new notion of deformation (*categorical deformation*) for a triangulated category, which is instead inspired by the second characterization. It turns out that, for categories, the two definitions do *not* coincide. The advantage of this new notion is that one can show that there exist exactly as many categorical deformations as one would expect – in this new setting, the curvature problem disappears. Of course, introducing a new notion brings its own problems. Namely, if A_ε is a deformation of an algebra A, the derived category $D(A_\varepsilon)$ is a classical deformation – and not a categorical deformation – of $D(A)$; to obtain a categorical deformation, one needs to appropriately "blow-up" the derived category $D(A_\varepsilon)$. This is the content of the other main part of this thesis, and is developed in Chapter 3: given a deformation A_ε of an algebra A, we introduce a "filtered derived category" $D^1(A_\varepsilon)$ which – unlike $D(A_\varepsilon)$ – is a categorical deformation of $D(A)$. This has the advantage of being well-defined also for so-called curved deformations of dg algebras, for which classical derived categories are unavailable. These two constructions fit together to form a particularly nice picture: given any deformation A_ε of an algebra A, the category $D^1(A_\varepsilon)$ is a categorical deformation of $D(A)$ which corresponds to the same cohomology class induced by the deformation A_ε (Theorem 5.3.2).

1.1 Overview of the problem

In this section we give a more detailed, but hopefully still accessible, introduction to the problem. As before, let k be a field and denote with $k[\varepsilon]$ the ring of dual numbers $k[t]/(t^2)$; unless otherwise stated, all deformations are assumed to be first order deformations, i.e. over $k[\varepsilon]$. A nice introduction to our main question comes from the deformation theory of abelian categories, introduced in [LVdB06; LvdB05]. Indeed, abelian categories can be considered as a (more elementary) model for noncommutative schemes [SVdB01]. Consider an algebra A over a field k, and a first order algebra deformation A_ε of A. Then the abelian category $\operatorname{Mod} A_\varepsilon$ of right A_ε-modules is a deformation, in an appropriate sense, of the abelian category $\operatorname{Mod} A$; this assignment defines a bijection

$$\operatorname{Def}_A(k[\varepsilon]) \xrightarrow{\sim} \operatorname{Def}^{\mathrm{ab}}_{\mathrm{Mod\,A}}(k[\varepsilon])$$

between the sets of abelian deformations of $\operatorname{Mod} A$ and the set of algebra deformations of A, both up to equivalence. Moreover algebra deformations of A are parametrized by the second Hochschild cohomology $\mathrm{HH}^2(A)$, and deformations of $\operatorname{Mod} A$ as an abelian category are parametrized by the second Hochschild cohomology $\mathrm{HH}^2_{\mathrm{ab}}(\operatorname{Mod} A)$. Using the existence of a natural isomorphism $\mathrm{HH}(A) \to \mathrm{HH}_{\mathrm{ab}}(\operatorname{Mod} A)$, one can show the existence of a commutative diagram of bijections

$$\begin{tikzcd}
\operatorname{Def}_A(k[\varepsilon]) \arrow[r] \arrow[d] & \mathrm{HH}^2(A) \arrow[d] \\
\operatorname{Def}^{\mathrm{ab}}_{\mathrm{Mod\,A}}(k[\varepsilon]) \arrow[r] & \mathrm{HH}^2_{\mathrm{ab}}(\operatorname{Mod} A)
\end{tikzcd} \tag{1.1}$$

where the left arrow associates to a deformation A_ε its abelian category of right modules.

Once the abelian theory was settled, a very natural question became whether and how that generalizes to the derived setting. A priori, one would expect the theory to admit a fairly straightforward extension. If one substitutes the algebra A for a dg algebra (denoted again with A), there is a natural notion of Hochschild cohomology for both A and its (enhanced) derived category $D(A)$; moreover, there exists a natural quasi-isomorphism $\mathbf{C}^\bullet(A) \overset{\chi_A}{\to} \mathbf{C}^\bullet(D(A))$ between the Hochschild complexes [Low08]. The ideal situation would hence be to have, as in the abelian case, a square of bijections

$$\begin{tikzcd}
\operatorname{Def}_A(k[\varepsilon]) \arrow[r] \arrow[d] & \mathrm{HH}^2(A) \arrow[d, "\chi_A"] \\
\operatorname{Def}^{\mathrm{tria}}_{D(A)}(k[\varepsilon]) \arrow[r] & \mathrm{HH}^2(D(A)),
\end{tikzcd}$$

where we have denoted with $\operatorname{Def}_A(k[\varepsilon])$ the set of dg algebra deformations of A and with $\operatorname{Def}^{\mathrm{tria}}_{D(A)}(k[\varepsilon])$ the set of deformations of $D(A)$ as a triangulated category. In particular, the left arrow should be defined by assigning to a deformation A_ε its derived category $D(A_\varepsilon)$. It was however soon observed that, at least in the naive sense, this picture could not hold.

1.1.1 The Hochschild complex and curved deformations

In the most basic sense, a (first order) deformation of an algebra A is an algebra structure on $A \otimes_k k[\varepsilon] \cong A \oplus \varepsilon A$ which reduces to A once the action of $t \in k[\varepsilon]$ is quotiented out. In the case of k-algebra A, the relationship between Hochschild 2-cocycles and deformations is transparent: an element $\mu \in \mathbf{C}^2(A)$ corresponds to a certain map $\mu \in \mathrm{Hom}_k(A \otimes_k A, A)$ which defines a deformation of A by deforming infinitesimally the multiplication map. If A is instead a dg algebra, then its Hochschild complex $\mathbf{C}^\bullet(A)$ is defined as the product totalization of the bicomplex

$$\mathbf{C}^{n,m}(A) = \mathrm{Hom}_k^n(A^{\otimes m}, A)$$

and a 2-cocycle $\mu \in \mathbf{C}^2(A)$ is given by a family

$$\{\mu_i\}_{i \in 0,1,\ldots},\ \mu_i \in \mathrm{Hom}_k^{2-i}(A^{\otimes i}, A).$$

In the case where the only nonzero components are μ_1 and μ_2, then one can still interpret the deformed object as a dg algebra; the multiplication m gets deformed into the star product $m_\varepsilon = m + t\mu_2$, while the differential d gets deformed to $d_\varepsilon = d + t\mu_1$. The condition $d\mu = 0$ then guarantees that the deformed object is still an (associative) dg algebra. However, as soon as a nonzero component $\mu_0 \in A^2$ is present, the just defined deformed object is not a dg algebra anymore; the deformed differential d_ε stops squaring to zero and A gets deformed into a *curved* algebra, a notion first introduced in [Pos93] in the context of Koszul duality. In the case where higher components μ_i for $i > 2$, are present, those induce infinitesimal operations that make the deformation A_ε into a cA_∞ (curved A_∞) algebra, a homotopical version of cdg algebras. Indeed, one verifies directly that there is a natural bijection between the set of Hochschild 2-cocycles and the set of cA_∞ deformations of a dg algebra [Low08].

This already leads to several issues: crucially, since the square of the differential of (modules) a curved algebra is not zero, there is no meaningful notion of derived category associated to it. This is significant, among other things, when trying to establish anything resembling the square (1.1); if a deformation induced by a certain Hochschild class has no associated derived category, then we have no way to construct from it a deformation of $D(A)$. Several objects approximating a notion of "derived category" for curved algebras have been constructed; most notably, the semiderived category and various derived categories of the second kind of Positselski, introduced in [Pos11; Pos18]. Despite these finding several applications in Koszul duality, they were shown in [KLN10] to vanish for certain deformations of dg algebras (see Example 3.2.2); hence, for deformation-theoretic applications a different type of invariant must be considered.

1.1.2 Morita deformations

It follows from the discussion above that, on the surface, dg deformations – deformations of a dg algebra that are still dg algebras – only capture a very small part of the Hochschild complex; that is, only the cocycles with $\mu_i = 0$ for $i \neq 1, 2$. The cocycles with 0-ary component give rise to curved deformations, and those with higher components to (curved, if μ_0 is also nonzero) A_∞ deformations. The situation improves significantly by considering a somewhat more general notion of deformation. Recall that if A, B are dg algebras, a dg A-B bimodule X is said to be a (derived) Morita equivalence if it induces equivalences between the respective derived categories. By the main result of [Kel03] any Morita A-B bimodule induces an equivalence between the Hochschild complexes $\mathbf{C}^\bullet(A)$ and $\mathbf{C}^\bullet(B)$; hence, this procedure allows to "transfer" a deformation of A to B and vice versa. Crucially, this equivalence does *not* preserve the product decomposition of the Hochschild complex. This means that one could have a curved deformation of A which, when transferred to B, corresponds to an uncurved deformation; similarly, $(c)A_\infty$ deformations can correspond to (c)dg deformations. In [KL09], Keller and Lowen defined a *Morita deformation* of A as a dg deformation B_ε of an algebra B equipped with an A-B Morita bimodule. Considering this wider class of deformations, one can indeed realize many more Hochschild classes via dg deformations. In [KL09], a natural map

$$\nu\colon \mathrm{Def}_A^{\mathrm{Mo}}(k[\varepsilon]) \to \mathrm{HH}^2(A)$$

was defined from the set of Morita deformations of an algebra to its second Hochschild cohomology by assigning to a Morita deformation B_ε the class obtained by transfer from the relevant class in $\mathrm{HH}^2(B)$. This already allows for a significant simplification: any class defining an A_∞ deformation can be obtained now as the image of a (dg, by definition) Morita deformation[2]. One might hope this to also work with curved deformations so that, up to changing the algebra A to a Morita equivalent one, every Hochschild class could be realized by a dg deformation. This was found not to be the case. Indeed, without further hypotheses, the map ν is neither injective nor, crucially, surjective; there exist explicit examples of Hochschild classes that cannot be produced via uncurved deformations, even after changing Morita representative. Hence, in a way, curved deformations are inescapable.

1.1.3 Deformations of triangulated categories

Up until this point we have only focused on the upper arrow of the diagram (1.1), and we have made no reference to what a deformation of a triangulated category ought to be. This is no coincidence, as the notion of deformation of a triangulated category is quite subtle. The classical way of looking at this question aims at abstracting the fact that a deformation over the ring $k[\varepsilon]$ of an algebraic object X corresponds to a $k[\varepsilon]$-linear object X_ε, reducing appropriately to X modulo t. For (enhanced, triangulated) categories, there are at least two meaningful notions of reduction; either the derived tensor product

$$- \otimes^{\mathrm{L}}_{k[\varepsilon]} k\colon k[\varepsilon]\text{-Cat} \to k\text{-Cat}$$

or the derived Hom

$$\mathbb{R}\operatorname{Hom}_{k[\varepsilon]}(k, -)\colon k[\varepsilon]\text{-Cat} \to k\text{-Cat}\,.$$

One can then define a deformation of a k-linear triangulated category $\mathcal{T}$ as a lift of $\mathcal{T}$ along either functor. Deformations using the first functor were considered by Lurie [Lur11] and incorporated in the general theory of formal moduli problems (that is, abstract deformation functors) [BKP18]. Deformations via the second functor were first considered by Lowen [Low08] in analogy with the abelian case [LVdB06], and then studied in more detail by Lowen-Van den Bergh and collaborators [GLVdB21; GLSVdB24]. In the following we will not distinguish between these two notions of deformations, calling both of them

[2] As is common in the setting of A_∞ structures, any A_∞ deformation can be rectified to a dg one.

classical deformations; in fact, despite – to the best of our knowledge – this not being explicitly discussed in the literature, it is reasonable that the two reduction functors (and hence, the two notions of deformations) are actually equivalent.

Classical deformation theory interacts well with classical derived categories, in the sense that if A_ε is a dg deformation of a dg algebra A, its derived category $D(A_\varepsilon)$ is a classical deformation of $D(A)$. However, in general, this does not lead to a well-behaved theory. At a formal level, the issue is that the functor that associates to a base ring R the set (really, space) of R-deformations of a category is not a deformation functor[3], and does not satisfy the formal properties that these functors usually enjoy. The conceptual issue is the same as for the case of algebras: the object that should (morally, but also technically in a sense that is made precise in [Lur11]) control the deformation theory of a category is its Hochschild complex. This however also contains curved deformations, and a curved deformation of an (enhanced) triangulated category is not in any natural way a triangulated category; while in the algebra case a curved deformation of a dg algebra is still, if anything, an algebra, a curved deformation of a category has no meaningful homotopical interpretation. This is hence another instance of the curvature problem.

1.1.4 The curvature problem

We can thus summarize the curvature problem into the following two (related) statements:

1. If A is a dg algebra, its Hochschild complex contains certain *curved* deformations; since those do not have a natural notion of (classical) derived category associated to them, there is no clear way to associate to a curved deformation of A a classical deformation of $D(A)$.
2. If $\mathcal{T}$ is a triangulated category, its classical deformations do not span the whole Hochschild complex.

There exist several approaches to this issue in the literature; those usually focus on finding more or less strict hypotheses on either the base or the object being deformed under which classical deformations do suffice to span the whole Hochschild complex; see for example [LVdB15; BKP18; Hai24] for the case of formal deformations of appropriately right-bounded objects, [LVdB12] for the case of schemes and [GLSVdB24] for deformations of dg categories with a nice t-structure. Concretely, in these approaches one somehow manages to find noncurved representatives for the relevant Hochschild classes. Nonetheless, it is known [KL09] that these nice results *cannot* hold unconditionally.

In the present work, we take a different approach. The principle here is to try to consider curved and uncurved deformations on the same level, which leads to defining a "derived category" of a deformation that is defined for curved and uncurved deformations alike. This in turn inspires a novel notion of *categorical deformation* for triangulated categories, allowing us to settle the curvature problem at least for first order deformations.

[3]We use the terms deformation functor and formal moduli problem interchangeably; the first is characteristic of the classical literature, while the second comes from the higher categorical treatment from [Lur11].

1.2 Content of the thesis

The thesis contains two main parts. In the first part, after introducing the needed background (Chapter 2), a categorical invariant of an arbitrary curved algebra deformation is constructed – called its n-derived category – and several formal properties are shown (Chapter 3). Then, in the second part, (Chapters 4 and 5) a notion of first order deformation for an enhanced dg category is developed; this is shown to have several desirable properties, and to encompass the case of the n-derived category (for $n = 1$) as a paradigmatic example.

1.2.1 The n-derived category

Let A be a dg algebra over k, and let A_n be a curved deformation of A over $R_n = k[t]/(t^{n+1})$; denote with $c \in A_n$ its curvature. Since for any cdg A_n-module M the square d_M^2 of the (pre)differential equals the action of c, in general curved modules have no cohomology. However, A_n is still a deformation of A and, in particular, its curvature must vanish modulo t; hence, $c \in tA_n$. It follows that $d_M^2(M) \subseteq tM$ and the graded pieces $t^iM/t^{i+1}M$ are all complexes. We thus define the n-derived category $D^n(A_n)$ as the quotient of the homotopy category of A_n-modules by the modules for which all graded pieces are acyclic. This category turns out be as well-behaved as one could hope. The first result we prove is the following (see Theorem 3.2.5 and Theorem 3.5.7 in the text):

Theorem 1.2.1. *The category $D^n(A_n)$ is compactly generated and allows for a semiorthogonal decomposition into $n+1$ copies of $D(A)$.*

This is obtained by describing $n+1$ explicit compact generators; the construction is quite similar, but a bit more general, to one that appeared in [LVdB15]. We use the existence of these generators to show (Theorem 3.4.3) that if A_n and B_n are cdg deformations of dg algebras A, B, then an A_n-B_n bimodule X_n induces an equivalence $D^n(A_n) \to D^n(B_n)$ if and only if the induced A-B bimodule $X = X_n \otimes_{R_n} k$ is a Morita bimodule – that is, induces an equivalence $D(A) \to D(B)$.

The semiorthogonal decomposition in Theorem 1.2.1 gives rise to a recollement (see the remark after Proposition 3.5.5)

$$\begin{tikzcd} D^{n-1}(A_{n-1}) \arrow[r, hook, "\iota_n"] & D^n(A_n) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\operatorname{Im} t^n"] & D(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} \tag{1.2}$$

in which the essential image of ι_n is given by the A_n-modules M for which t^nM is acyclic.

In the case where the deformation A_n has no curvature and one can speak of the classical derived category, we show the following relation between the classical and n-derived category (Corollary 3.1.10):

Theorem 1.2.2. *There exists a localization functor*

$$D^n(A_n) \to D(A_n)$$

admitting both a left and a right adjoint.

As a corollary, we obtain the existence of a fully faithful functor $D(A_n) \hookrightarrow D^n(A_n)$.

The category of A_n-modules allows for an explicit construction of semifree (and a fortiori n-homotopy projective) resolutions (see Section 3.3), and in fact has a "projective" model structure (see Section 3.7). These results are obtained along the lines of [Kel94] and [CH02] respectively. It also has homotopy injective resolutions, the construction of which is detailed in the Section 3.9.

In 3.8, we extend most of our results to the formal setting of a curved $k[[t]]$-deformation A_t of A by introducing the *t-derived category of torsion modules* $D^t(A_t)^{\mathrm{tor}}$ (Definition 3.8.4), which constitutes an ∞-categorical colimit of the n-derived categories.

The final goal of this section is to compare our constructions with Positselski's semiderived category. We show (Corollaries 3.6.8 and 3.8.6):

Theorem 1.2.3. *There are admissible embeddings $D^{\mathrm{si}}(A_n) \to D^n(A_n)$ in the infinitesimal setting, and a left admissible embedding $D^{\mathrm{si}}(A_t^{\mathrm{co}}) \to D^t(A_t)^{\mathrm{tor}}$ in the formal setting.*

It is a natural question to ask how the definition of the n-derived category generalizes to deformations over local artinian rings which have dimension higher than 1. The obvious proposal is to simply quotient out from the homotopy category the modules which are filtered acyclic; in the local artinian setting, this corresponds to considering the $\mathfrak{m}$-adic filtration, where $\mathfrak{m}$ is the maximal ideal. It turns out however that for most technical results it is very important that the ideal defining the filtration is principal, and without this assumption many arguments break down; hence to tackle the general case, different techniques will be needed. Let us conclude with a technical point. In [KLN10], the vanishing of several "derived" categories of cdg algebras was shown assuming only some very basic axioms, key among which was the fact that any short exact sequence should give rise to a triangle in the quotient. This is the property that we negate in order to obtain a derived category that is guaranteed not to vanish. Indeed, in our case a short exact sequence of A_n-modules $0 \to M \to N \to L \to 0$ is a triangle precisely when for all $i = 1, \ldots, n$ the induced sequence

$$0 \to \frac{M}{t^i M} \to \frac{N}{t^i N} \to \frac{L}{t^i L} \to 0$$

is exact. Vice versa, all triangles up to isomorphism are of this form.

1.2.2 Deformations of triangulated categories

The goal of Chapters 4 and 5 is to answer the question of in what sense the n-derived category is a deformation of the category $D(A)$; indeed, it does not appear to be a classical deformation. In Section 3.3.9 we introduce a notion of *categorical deformation* of an (enhanced) triangulated category, and show that (first order) deformations are parametrized by the second Hochschild cohomology group; in Section 5 we show that the 1-derived category of a first order algebra deformation A_ε is a categorical deformation of $D(A)$, and investigate the corresponding Hochschild class. Since in this section we will only be interested in first order deformations, we will denote with A_ε a first order algebra deformation, and with $D^\varepsilon(A_\varepsilon)$ its 1-derived category $D^1(A_1)$.

The novel idea is to try to categorify the classical square zero extension

$$0 \to A \to A_\varepsilon \to A \to 0$$

associated to a first order algebra deformation. Roughly speaking, a (first order) categorical deformation of $\mathcal{T}$ consists of a triangulated category $\mathcal{T}_\varepsilon$ together with two "extensions": a recollement

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i"] & \mathcal{T}_\varepsilon \arrow[l, bend right, "Q"'] \arrow[l, bend left, "K"] \arrow[r, "E"] & \mathcal{T} \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd} \tag{1.3}$$

together with a Yoneda 2-extension of functors that can be informally depicted as

$$0 \to E \xrightarrow{\delta_1} K \xrightarrow{\alpha} Q \xrightarrow{\delta_2} E \to 0.$$

One can easily obtain a Hochschild class from such a deformation, by composing the latter extension with the functor G and taking the associated class in

$$\mathrm{Ext}^2_{\mathrm{Fun}(\mathcal{T},\mathcal{T})}(EG, EG) \cong \mathrm{Ext}^2_{\mathrm{Fun}(\mathcal{T},\mathcal{T})}(\mathrm{id}_\mathcal{T}, \mathrm{id}_\mathcal{T}) \cong \mathrm{HH}^2(\mathcal{T}).$$

Conversely, for an arbitrary Hochschild class in $\mathrm{HH}^2(\mathcal{T})$ represented by

$$\mathrm{id}_\mathcal{T}[-1] \to \mathrm{id}_\mathcal{T}[1]$$

the corresponding deformation $\mathcal{T}_\varepsilon$ is constructed by gluing – in the sense of [KL12] – two copies of $\mathcal{T}$ along the cone of the 2-class. Using this construction, one can show (Theorem 4.1.9):

Theorem 1.2.4. *There exists a bijection between $\mathrm{HH}^2(\mathcal{T})$ and the set $\mathrm{CatDef}_\mathcal{T}(k[\varepsilon])$ of equivalence classes of categorical deformations of $\mathcal{T}$.*

We also show compatibility with the 1-derived category. In particular, recalling the existence of the natural map

$$\chi_A\colon\ \mathrm{HH}(A) \to \mathrm{HH}(D(A)),$$

we show (Theorem 5.1.1):

Theorem 1.2.5. *Let A be a dg algebra, and A_ε a cdg deformation of A corresponding to a Hochschild cocycle $\mu_A \in \mathbf{C}^2(A)$. Then the category $D^\varepsilon(A_\varepsilon)$ is a categorical deformation of $D(A)$, and the class $\mu_{D(A)}(D^\varepsilon(A_\varepsilon)) \in \mathrm{HH}^2(D(A))$ coincides with $\chi_A(\mu_A)$.*

At this point, we prove a fact that until now has been implicitly assumed in this introduction: if one allows for curved Morita deformations of a dg algebra A – that is, cdg deformations B_n of an algebra B equipped with a Morita equivalence with A – then Morita deformations are indeed parametrized by Hochschild cohomology. More precisely, we show (Theorem 5.2.1):

Theorem 1.2.6. *Let* $\mathrm{cDef}_A(k[\varepsilon])$ *denote the set of equivalence classes of curved Morita deformations of a dg algebra* A. *There exists a bijection*

$$\nu\colon \mathrm{cDef}_A(k[\varepsilon]) \to \mathrm{HH}^2(A).$$

Here, the most subtle part is correctly defining what an equivalence of deformations ought to look like, since the classical notions break down; this turns out to reduce to finding an appropriate notion of cofibrancy for curved bimodules (see Definiton 5.2.2).

It should be pointed out that there is a qualitative difference between the *curved deformations* of the base algebra A and the *categorical deformations* of its derived category $D(A)$; indeed, while the base algebra A might have deformations that are curved – and thus live outside of the realm of classical homological algebra – a categorical deformation of $D(A)$ is *always* a triangulated category – by definition. This is one of the main novelties of the notion of categorical deformation: given an arbitrary Hochschild cocycle, even one that has nontrivial curvature component, our approach allows to construct a deformation which is again a dg category. This situation makes sense from the perspective of noncommutative algebraic geometry; a noncommutative space can be represented by different algebraic models: small ones – for example, a dg algebra – and large ones – for example, its derived category. The point is that small and large models have qualitatively different deformation theories. While for a small model there is no escaping curvature – there exist algebra deformations that are intrinsically curved – the large models offer enough flexibility to realize every deformation without leaving the world of triangulated categories. To obtain this, however, one must part with the idea that small models and large models should correspond via classical (or, for that matter, second kind) derived categories.

Finally, we can state and prove the main result of this thesis (Theorem 5.3.2).

Theorem 1.2.7. *Let* A *be a dg algebra. Then for any curved deformations* A_ε *of* A, *its* 1*-derived category is a categorical deformation of* $D(A)$ *and there is a commutative diagram of bijections*

$$\begin{tikzcd} \mathrm{cDef}_A(k[\varepsilon]) \arrow[r, "\nu"] \arrow[d, "D^1(-)"'] & \mathrm{HH}^2(A) \arrow[d, "\chi_A"] \\ \mathrm{CatDef}_{D(A)}(k[\varepsilon]) \arrow[r, "\mu_{D(A)}"] & \mathrm{HH}^2(D(A)) \end{tikzcd} \tag{1.4}$$

The existence of this diagram leads us to consider the curvature problem "solved" for first order deformations.

After that, in Section 5.4 we discuss an interesting relation between categorical deformations and categorical resolution of singularities in the sense of [KL12]. The key observation is that if $\mathcal{T}$ is a smooth dg category, then any deformation $\mathcal{T}_\varepsilon$ is also smooth. Hence, if A_ε is a deformation of a homologically smooth dg algebra A, its 1-derived category $D^\varepsilon(A_\varepsilon)$ is still smooth. On the other hand hand, even if A_ε has no curvature, the derived category $D(A_\varepsilon)$ is *never* smooth – essentially because of the nilpotent element t. Using Theorem 1.2.7, one can conclude that $D^1(A_\varepsilon)$ is a categorical resolution of singularities of $D(A_n)$, in the sense that it is a smooth dg category that $D(A_n)$ embeds into. This leads to considering $D^\varepsilon(A_\varepsilon)$ as a *blowup* of $D(A_\varepsilon)$. Finally, in Chapter 6 we discuss some potential future developments.

Relation to other work

The content of this thesis is the (mostly disjoint) union of the three papers [LL24], [Leh24] and [LL25]. The main differences lie in the order in which some of the topics are introduced, as well as in some discussion that has been added. Specifically, the results of Chapter 3 are contained in [LL24], and those of Chapters 4 and 5 are contained in [LL25]. The only exceptions are given by in Sections 3.4 and 5.2, which are instead contained in [Leh24].

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Chapter 2

Background

We begin by recalling all the technical tools that will be used throughout the text, as well as by fixing some notation. Nothing in this chapter is particularly new. The only sections that are not entirely recollections of known facts are Section 2.1.4 which deals with Yoneda extensions in triangulated categories and Section 2.3.3, which defines recollements of quasi-functors.

Conventions. All graded objects are assumed to be graded by the integers; we employ cohomological grading, i.e. differentials increase the degree. All rings and algebras are associative and unital. If $\mathcal{C}$ is a dg category, we will denote by $Z^0\mathcal{C}$ it's underlying category, and with $H^0\mathcal{C}$ its homotopy category; the complex of morphisms in the category $\mathcal{C}$ will be denoted $\mathrm{Hom}_{\mathcal{C}}(X, Y)$; in the case of modules over an algebra, we will denote with $\mathrm{Hom}_A(M, N)$ the complex of morphisms from M to N. If $\mathcal{T}$ is a triangulated category linear over a ring R, the notation $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ will denote the R-module of morphisms in $\mathcal{T}$. At times, we will also use the notation $\mathrm{Ext}^i_{\mathcal{T}}(A, B)$ to signify $\mathrm{Hom}_{\mathcal{T}}(A, B[n])$. We will denote with k a fixed base field, and with $k[\varepsilon]$ the k-algebra $k[t]/(t^2)$. Unless otherwise specified, deformation will mean infinitesimal first order deformation, i.e. deformation over $k[\varepsilon]$.

2.1 Triangulated categories and derived categories

We assume the reader is familiar with the basic notions and properties of triangulated categories. In this section we give some relevant definitions and list the results that will be explicitly cited later in the text.

2.1.1 Localizations of triangulated categories

Let $\mathcal{T}$ be a triangulated category admitting small coproducts and $\mathcal{S} \subseteq \mathcal{T}$ a thick subcategory, i.e. a triangulated subcategory closed under retracts. The subcategory ${}^{\perp}\mathcal{S}$ is defined as the full subcategory of all objects X such that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all $Y \in \mathcal{S}$; analogously, $\mathcal{S}^{\perp} \subseteq \mathcal{T}$ is given by the objects Y for which $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all $X \in \mathcal{S}$. We will denote by $\mathcal{T}/\mathcal{S}$ the Verdier quotient of $\mathcal{T}$ by $\mathcal{S}$. The following result is classical, see for example [Kra09, Lemma 4.8.1, Proposition 4.9.1] for a proof.

Proposition 2.1.1. *Let $X, Y \in \mathcal{T}$. If either $X \in {}^{\perp}\mathcal{S}$ or $Y \in \mathcal{S}^{\perp}$, then the natural map*

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$$

is an isomorphism. Moreover, the following are equivalent:

- *The quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a left adjoint, which is automatically fully faithful;*
- *for any $X \in \mathcal{T}$, there exists a triangle in $\mathcal{T}$*

$$P \to X \to S \to P[1]$$

  with $P \in {}^{\perp}\mathcal{S}$ and $S \in \mathcal{S}$.
- *The composition*

$${}^{\perp}\mathcal{S} \hookrightarrow \mathcal{T} \to \mathcal{T}/\mathcal{S}$$

  is an equivalence.

The left adjoint to the quotient functor is constructed by associating to the object X the object P as per the triangle above. As a consequence of this construction, the essential image of the left adjoint is the subcategory ${}^{\perp}\mathcal{S} \subseteq \mathcal{T}$. The dual statement about the existence of a right adjoint also holds with $\mathcal{S}^{\perp}$ in place of ${}^{\perp}\mathcal{S}$ and the arrows in the triangle reversed.

2.1.2 Semiorthogonal decompositions

A *semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{S}_0, \mathcal{S}_1 \rangle$$

of a triangulated category $\mathcal{T}$ is by definition given by a pair of triangulated subcategories $\mathcal{S}_0, \mathcal{S}_1 \subseteq \mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(S_1, S_0) = 0$ for all $S_0 \in \mathcal{S}_0$ and $S_1 \in \mathcal{S}_1$ and such that for every $X \in \mathcal{T}$ there exists a triangle

$$S_1 \to X \to S_0 \to S_1[1]$$

with $S_0 \in \mathcal{S}_0$ and $S_1 \in \mathcal{S}_1$; the definition generalizes straightforwardly to decompositions having more that one factor. A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is said to be *left admissible* if the inclusion admits a left adjoint, *right admissible* if the inclusion admits a right adjoint and *admissible* if it is both left and right admissible; any right admissible subcategory $\mathcal{S}$ defines a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{S}^{\perp}, \mathcal{S} \rangle$$

and dually for left admissible subcategories.

2.1.3 Compactly generated triangulated categories

If $\mathcal{T}$ is a triangulated category admitting small coproducts, an object $C \in \mathcal{T}$ is said to be compact if $\mathcal{T}(C, -)$ commutes with small coproducts. The category $\mathcal{T}$ is said to be *compactly generated* if there exists a set $\mathcal{C} \subseteq \mathcal{T}$ of compact objects generating $\mathcal{T}$, i.e. such that for all $X \in \mathcal{T}$ one has

$$X = 0 \Leftrightarrow \mathrm{Hom}_{\mathcal{T}}(C[n], S) = 0 \text{ for all } n \in \mathbb{Z} \text{ and all } C \in \mathcal{C}.$$

A triangulated subcategory is said to be localizing if it is closed under small coproducts. It can be proven ([SS03, Lemma 2.2.1]) that, provided that $\mathcal{C} \subseteq \mathcal{T}$ is composed of compact objects, the condition that $\mathcal{C}$ generates $\mathcal{T}$ is equivalent to the fact that $\mathcal{T}$ coincides with the minimal localizing subcategory containing $\mathcal{C}$.

2.1.4 Yoneda extensions in triangulated categories

Let $\mathcal{T}$ be a triangulated category. For future use, we will employ the notation $\mathrm{Ext}^n(A, B)$ to denote the k-module of morphisms in $\mathcal{T}$ from A to $B[n]$.

Definition 2.1.2. An *(Yoneda) n-extension* $\mathcal{E}$ from A to B consists of n exact triangles

$$C_i \to E_i \to C_{i+1} \to C_i[1] \tag{2.1}$$

for $i = 0, \dots, n-1$ with $C_0 = B$ and $C_n = A$. The *splicing sequence* of the given n-extension is the resulting sequence of morphisms

$$\sigma(E) = (0 \to B \to E_0 \to E_1 \to \cdots \to E_{n-1} \to A \to 0). \tag{2.2}$$

For another n-extension $\mathcal{E}' = (C'_i \to E'_i \to C_{i+1} \to C'_i[1])$, a *morphism of n-extensions* $\mathcal{E} \to \mathcal{E}'$ consists of maps $g_i : C_i \to C'_i$ and $f_i : E_i \to E'_i$ with $g_0 = 1_B$, $g_n = 1_A$ inducing morphisms of all the relevant triangles.

An n-extension $E = (C_i \to E_i \to C_{i+1} \to C_i[1])$ determines n connecting morphisms

$$\varphi_i : C_{i+1} \to C_i[1]. \tag{2.3}$$

The *Ext-class* of $\mathcal{E}$ is by definition the composition

$$\varphi(\mathcal{E}) = \varphi_0[n-1]\varphi_1[n-2] \dots \varphi_{n-1} : A \to B[n]. \tag{2.4}$$

Let $\mathrm{Ext}^n_Y(A, B)$ denote the set of n-extensions from A to B. Consider the map

$$\varphi : \mathrm{Ext}^n_Y(A, B) \to \mathrm{Ext}^n(A, B) : \mathcal{E} \mapsto \varphi(\mathcal{E}). \tag{2.5}$$

The map φ is readily seen to be surjective, as in fact any factorization of a morphism $A \to B[n]$ into an n-simplex as in (2.4) allows for the construction of a corresponding pre-image.

Proposition 2.1.3. *Consider n-extensions $\mathcal{E}$, $\mathcal{E}'$ from A to B. If there exists a morphism $\mathcal{E} \to \mathcal{E}'$, then we have $\varphi(\mathcal{E}) = \varphi(\mathcal{E}')$.*

Proof. This easily follows from the definition of φ and the requirement that a morphism fixes both A and B. $\square$

Remark. Let $\Sigma^n(A,B)$ denote the set of sequences of shape (2.2) in which the composition of every two consecutive maps is zero. Unlike in the familiar case of extensions in an abelian category, the map $\sigma : \operatorname{Ext}^n_Y(A,B) \to \Sigma^n(A,B)$ is in general neither surjective nor injective, due to the fact that cones are merely weak cokernels and the map towards the cone fails to be an epimorphism in general.

2.1.5 Derived categories of dg algebras

Let A be a dg algebra. We will denote by $\operatorname{Mod} A$ the dg category of right A-modules and with $\operatorname{Hot}(A)$ the homotopy category $H^0(\operatorname{Mod} A)$. The category $\operatorname{Hot}(A)$ has the structure of a triangulated category, with triangles given by graded split short exact sequences i.e. short exact sequence that split as sequences of graded A-modules. The derived category $D(A)$ is the quotient of $\operatorname{Hot}(A)$ by the subcategory $\mathrm{Ac} \subseteq \operatorname{Hot}(A)$ given by the acyclic A-modules. An A-module M is said to be *homotopy projective* if it lies in ${}^{\perp}\mathrm{Ac}$ and *homotopy injective* if it lies in $\mathrm{Ac}^{\perp}$; the quotient $\operatorname{Hot}(A) \to D(A)$ admits both a left adjoint $\mathbf{p}$ and a right adjoint $\mathbf{i}$ which are fully faithful and whose essential images are given respectively by the homotopy projective and homotopy injective modules. The derived category is always compactly generated; we include the elementary proof of this fact since it will serve as a blueprint for the proof in the curved case.

Proposition 2.1.4. *The free module $A \in \operatorname{Mod} A$ is a homotopy projective compact generator of $D(A)$.*

Proof. Recall the isomorphism

$$\operatorname{Hom}_A(A, M) \cong M$$

for any A-module M; if M is acyclic it's clear that

$$\operatorname{Hom}_{\operatorname{Hot}(A)}(A[n], M) \cong H^{-n}M = 0$$

so A is homotopy projective and $\operatorname{Hom}_{\operatorname{Hot}(A)}(A,-)$ computes $\operatorname{Hom}_{D(A)}(A,-)$. Therefore, if M is such that $\operatorname{Hom}_{D(A)}(A[n], M)$ for all n, it follows that

$$\operatorname{Hom}_{\operatorname{Hot}(A)}(A[n], M) \cong H^{-n}M = 0$$

and M is acyclic. Finally, A is compact since taking cohomology commutes with coproducts. $\square$

Let A, B be dg algebras and X a dg A-B bimodule. The bimodule X induces an adjunctions

$$\operatorname{Mod} A \underset{-\otimes_A X}{\overset{\operatorname{Hom}_B(X,-)}{\rightleftarrows}} \operatorname{Mod} B.$$

The functors $\operatorname{Hom}_B(X,-)$ and $-\otimes_A X$ can be derived by precomposing respectively with the functors $\mathbf{i}$ and $\mathbf{p}$, giving rise to an adjoint pair

$$D(A) \underset{-\otimes^{\mathbb{L}}_A X}{\overset{\mathbb{R}\operatorname{Hom}_B(X,-)}{\rightleftarrows}} D(B). \tag{2.6}$$

The bimodule X is said to be a (derived) Morita equivalence if the adjunction (2.6) is an equivalence.

2.1.6 On enhancements

We will work with a specific model for the (enhanced) derived category $D(A)$, given by the category $\mathrm{Tw}(A)$ of (one-sided) *twisted complexes*. An object of this category is given by an ordinal I and a pair

$$M = (\oplus_{i\in I} A[n_i], \{f_{ij}\}_{i,j\in I})$$

where $f_{ij} \in A[n_i - n_j]$ have the property that $f_{ij} = 0$ for $i \leq j$ and $df + f^2 = 0$ (see e.g. [BLL04; Low08] for more details). The Hom-complex between two twisted object is given by the appropriate space of matrices, with twisted differential. The object M can be seen as the graded A-module $\oplus_i A[n_i]$ with differential given by $d + f$. There exists a fully faithful totalization dg functor

$$\mathrm{Tw}(A) \to \mathrm{Mod}\, A$$

which sends a twisted complex to the just described A-module. Its essential image is given by the semi-free A-modules, hence $\mathrm{Tw}(A)$ gives a dg enhancement of $D(A)$. In the following, we will not distinguish between $D(A)$ and its enhancement $\mathrm{Tw}(A)$.

2.2 Curved algebras and deformations

We can now introduce one of the central objects in this thesis: curved algebras and, more specifically, curved deformations of dg algebras.

2.2.1 Curved algebras

Let R be a commutative ring.

Definition 2.2.1 ([Pos18])**.** A *cdg algebra* $\mathcal{A}$ over R is given by a graded R-algebra $\mathcal{A}^\#$ equipped with a derivation $d_\mathcal{A} \in \mathrm{Hom}_R(\mathcal{A}^\#, \mathcal{A}^\#)$ of degree 1 and an element $c \in A^2$ such that $d_\mathcal{A}(c) = 0$ and $d_\mathcal{A}^2 = [c, -]$.

The derivation $d_\mathcal{A}$ is called predifferential and the element c curvature of the algebra. Any dg algebra is in a natural way a cdg algebra by letting $c = 0$.

Definition 2.2.2. A left *cdg module* over a cdg algebra $\mathcal{A}$ is a graded left $\mathcal{A}^\#$-module $M^\#$ equipped with an $\mathcal{A}$-derivation $d_M \in \mathrm{Hom}_R(M^\#, M^\#)^1$ such that $d_M^2 m = cm$ for all $m \in M$. A right cdg $\mathcal{A}$-module is a right graded $\mathcal{A}^\#$-module N with a derivation $d_N \in \mathrm{Hom}_R(N, N)^1$ such that $d_N^2 n = -nc$ for all $n \in N$. If $\mathcal{B}$ is another cdg algebra, a cdg $\mathcal{A}$-$\mathcal{B}$ bimodule X is a graded $\mathcal{A}^\#$-$\mathcal{B}^\#$ bimodule $X^\#$ equipped with a degree 1 map $d_X : X^\# \to X^\#$ compatible with the differentials of $\mathcal{A}$ and $\mathcal{B}$ and with the property that $d_X^2 x = c_\mathcal{A} x - x c_\mathcal{B}$ for all $x \in X$.

Crucially, as soon as the curvature is not zero, the algebra is not a module over itself; the nonexistence of free modules is one of the main features of our categories of curved modules.

Definition 2.2.3. If M and N are cdg $\mathcal{A}$-modules, the complex $\operatorname{Hom}_{\mathcal{A}}(M,N)$ of $\mathcal{A}$-linear morphisms is defined in the same way as the complex of morphisms between dg modules: it has in degree n the R-module of $\mathcal{A}^{\#}$-module morphisms of degree n, and for $f \in \operatorname{Hom}_{\mathcal{A}}(M,N)^n$ differential defined as $(df)m = d_N(fm) - (-1)^n f(d_M m)$. Even if M and N are not themselves complexes, the space of morphisms is one (i.e. satisfies $d^2 = 0$). The dg category $\operatorname{Mod}\mathcal{A}$ has as objects the cdg right $\mathcal{A}$-modules and the just defined complex of morphisms as hom-complex; composition is defined in the obvious way. The same definitions work for right modules, giving rise to the dg category $\mathcal{A}\operatorname{Mod}$ of left cdg $\mathcal{A}$-modules.

To simplify the notation, we write $\operatorname{Hom}_{\mathcal{A}}(M,N)$ instead of $\operatorname{Hom}_{\operatorname{Mod}\mathcal{A}}(X,Y)$ for the complex of morphisms inside the dg category $\operatorname{Mod}\mathcal{A}$. From now on, all modules, unless otherwise specified, will be assumed to be right modules.

Example 2.2.1 (Matrix factorizations)**.** Let A be any R-algebra and f an element in its center. Consider the graded algebra $A[u,u^{-1}]$ with u in degree 2; this can be made into a cdg algebra $\mathcal{B}$ with zero differential and curvature $c_{\mathcal{B}} = fu$. Then by definition, a cdg $\mathcal{B}$-module M is given by a 2-periodic graded object

$$\ldots \to M_0 \overset{d_0}{\to} M_1 \overset{d_1}{\to} M_0 \overset{d_0}{\to} M_1 \to \ldots$$

such that $d_i d_j = f \operatorname{id}_{M_i}$. These are the objects that are usually called *matrix factorizations* for f, with the caveat that the modules appearing in matrix factorizations are usually assumed to be projective. For more about this construction, see e.g. [Dyc11] or [EP15] for the geometric case.

The category $\operatorname{Mod}\mathcal{A}$ is a pretriangulated dg category; we will denote by $\operatorname{Hot}(\mathcal{A})$ the triangulated category $H^0(\operatorname{Mod}\mathcal{A})$. Like in the classical case, triangles in $\operatorname{Hot}(\mathcal{A})$ are the short exact sequences that split at the graded level.

2.2.2 Deformations

Let A be a dg algebra over k. Denote by R_n the commutative k-algebra $k[t]/(t^{n+1})$.

Definition 2.2.4. A *cdg deformation* A_n of A of order n is the datum of a structure of a cdg algebra on the R_n-module $A \otimes_k R_n \cong A[t]/(t^{n+1})$ which reduces modulo t to the dg algebra structure of A.

Alternatively, as usual one may define a cdg deformation of A of order n to be an R_n-free cdg algebra A_n equipped with a dg algebra isomorphism $A_n \otimes_{R_n} k \cong A$. For our purposes the slightly more restrictive definition we gave suffices, but everything we show can be readily carried over to the more general setting.

Remark. Usually in deformation theory one considers deformations over arbitrary local artinian algebras, of which the algebras R_n are one example. However the general case makes the theory developed in this paper become significantly more complicated, and we will limit ourselves to the already interesting case of R_n.

We will denote by c the curvature of A_n and with d_{A_n} its predifferential. The multiplication of A_n will be denoted with the juxtaposition.

Some adjunctions

For all $0 \leq m \leq n$, denote by A_m the R_m-algebra

$$A_n \otimes_{R_n} R_m \cong \frac{A_n}{t^{m+1}A_n},$$

where the R_n-action on R_m is defined via the natural surjection $R_n \to R_m$; we will say that A_m is the deformation of order m of A induced by A_n.

The surjection $R_n \to R_m$ induces a surjection $A_n \to A_m$, defining via restriction of scalars the fully faithful functor

$$F\colon \operatorname{Mod} A_m \to \operatorname{Mod} A_n.$$

This has a left adjoint

$$\begin{aligned}\operatorname{Coker} t^{m+1}\colon \operatorname{Mod} A_n &\to \operatorname{Mod} A_m\\ M &\to \frac{M}{t^{m+1}M} \cong A_m \otimes_{A_n} M\end{aligned}$$

and a right adjoint

$$\begin{aligned}\operatorname{Ker} t^{m+1}\colon \operatorname{Mod} A_n &\to \operatorname{Mod} A_m\\ M &\to \operatorname{Ker} t_M^{m+1} \cong \operatorname{Hom}_{A_n}(A_m, M),\end{aligned}$$

where we have denoted with t_M the action of t on M.

2.3 Higher categories and quasi-functors

We now turn to the higher categorical aspects of the theory. We employ the model of dg categories, for which we refer the reader to [Kel06] for the basic notions. When we talk about enhanced triangulated categories, we will always mean pretriangulated dg categories. As models for ∞-functors between dg categories we use quasi-functors; Recall that if $\mathcal{A}, \mathcal{B}$ are dg categories, a *quasi-functor* is an object $X \in D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ which is right quasi-representable, i.e. such that for every $A \in \mathcal{A}$, the element $X(-, A)$ is quasi-isomorphic to a representable $\mathcal{B}$-module; a quasi-functor is said to be a quasi-equivalence if it admits an inverse; this is equivalent to inducing an equivalence between the respective homotopy categories.

Quasi-functors will play a crucial role in Chapter 4, where we will use several constructions involving them; we list here some of those, as well as some of the relevant properties.

2.3.1 General operations

Despite quasi-functors not being literal functors, we will still use classical category-theoretical notations for various constructions. For simplicity, we will write $\mathcal{A} \xrightarrow{F} \mathcal{B}$ to denote a quasi-functor $F \in D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be dg categories. Given two quasi-functors $\mathcal{A} \xrightarrow{G} \mathcal{B}$ and $\mathcal{B} \xrightarrow{F} \mathcal{C}$, their composition $G \otimes^{\mathrm{L}}_{\mathcal{B}} F$ will be denoted with FG. Given $F, G \colon \mathcal{B} \to \mathcal{C}$, $M, N \colon \mathcal{A} \to \mathcal{B}$, $f \in \mathrm{Hom}_{D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})}(M, N)$ and $\eta \in \mathrm{Hom}_{D(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{C})}(F, G)$, we will denote with ηM the morphism $\mathrm{id}_M \otimes^{\mathrm{L}} \eta \in \mathrm{Hom}_{D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{C})}(FM, GN)$ and with Ff the morphism $f \otimes^{\mathrm{L}} \mathrm{id}_F \in \mathrm{Hom}_{D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{C})}(FM, GN)$. This is compatible with the classical use, as the following lemma shows.

Lemma 2.3.1. *The diagram*

$$\begin{tikzcd} FM \arrow[r, "\eta M"] \arrow[d, "Ff"'] & GM \arrow[d, "Gf"] \\ FN \arrow[r, "\eta N"] & GN \end{tikzcd}$$

commutes.

Proof. Writing down the definitions, both compositions are seen to equal $f \otimes^{\mathrm{L}} \eta$. □

2.3.2 Adjunctions

We will use extensively the notion of adjunction of quasi-functors, see [Gen17]. Given two quasi-functors $\mathcal{A} \xrightarrow{F} \mathcal{B}$ and $\mathcal{B} \xrightarrow{G} \mathcal{A}$, we will say [Gen17, Definition 6.2] that F is left adjoint to G if there exist two morphisms $\mathrm{id}_{\mathcal{A}} \to FG$ in $D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{A})$ and $GF \to \mathrm{id}_{\mathcal{B}}$ in $D(\mathcal{B}^{\mathrm{op}} \otimes \mathcal{B})$ satisfying the usual unit-counit equation in the appropriate derived category. We will frequently need the following statement, which is straightforward to prove:

Proposition 2.3.2. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be dg categories and $\mathcal{A} \xrightarrow{F} \mathcal{B}$ a quasi-functor with right adjoint G. Let $\mathcal{C} \xrightarrow{M} \mathcal{A}$ and $\mathcal{C} \xrightarrow{N} \mathcal{B}$. There is a natural isomorphism*

$$\mathrm{Hom}_{D(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{B})}(FM, N) \cong \mathrm{Hom}_{D(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{A})}(M, GN).$$

Note that it follows from [DKSS24] that this notion of adjunction is compatible with the one for the associated ∞-functors between stable ∞-categories.

2.3.3 Recollements of quasi-functors

Let $\mathcal{A}$, $\mathcal{B}$ be pretriangulated dg categories. A categorical extension of $\mathcal{A}$ by $\mathcal{B}$, or recollement, is a dg category $\mathcal{C}$ equipped with two quasi-functors

$$\mathcal{A} \xrightarrow{i} \mathcal{C} \xrightarrow{E} \mathcal{B}$$

satisfying the following conditions:

1. The composition $E \circ i$ vanishes;
2. The quasi-functor i admits a left adjoint Q and a right adjoint K; the quasi-functor E admits a left adjoint G. The adjunctions are induced by units

$$\mathrm{id}_{\mathcal{C}} \xrightarrow{q} iQ, \ \mathrm{id}_{\mathcal{C}} \xrightarrow{g} EG, \ \mathrm{id}_{\mathcal{A}} \to Ki$$

and counits

$$Qi \to \mathrm{id}_{\mathcal{A}}\,,\ iK \xrightarrow{k} \mathrm{id}_{\mathcal{C}}\,,\ GE \xrightarrow{\xi} \mathrm{id}_{\mathcal{C}}\,.$$

3. The transformations

$$\mathrm{id}_{\mathcal{A}} \to Ki,\ Qi \to \mathrm{id}_{\mathcal{A}} \text{ and } \mathrm{id}_{\mathcal{C}} \xrightarrow{g} EG$$

are isomorphisms in the appropriate derived categories; this corresponds to the statement that i and G are homotopically fully faithful.

4. The square

$$\begin{tikzcd} GE \arrow[r, "\xi"] \arrow[d] & \mathrm{id}_{\mathcal{C}} \arrow[d, "q"] \\ 0 \arrow[r] & iQ \end{tikzcd}$$

is homotopy cartesian in $D(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$, i.e. there exists a natural transformation $iQ \xrightarrow{\partial} GE[1]$ fitting into a triangle

$$GE \xrightarrow{\xi} \mathrm{id}_{\mathcal{C}} \xrightarrow{q} iQ \xrightarrow{\partial} GE[1];$$

this corresponds to the classical fact that any object of $\mathcal{C}$ can be obtained as an extension of an object of $\mathcal{A}$ by an object of $\mathcal{B}$.

If these conditions are satisfied, then I also admits a right adjoint, which we will leave unlabeled. Pictorially, we draw

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"] & \mathcal{C} \arrow[l, bend right, "Q"'] \arrow[l, bend left, "K"] \arrow[r, "E"] & \mathcal{B}. \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

Remark. It is easy to show that a recollement of quasi-functors gives rise to a semiorthogonal decomposition

$$H^0\mathcal{C} = \langle H^0 i\mathcal{A}, H^0 G\mathcal{B}\rangle$$

of the homotopy category; conditions (1) and (2) imply that the two subcategories are indeed semiorthogonal, while condition (4) implies that any object of $H^0\mathcal{C}$ can be obtained as the cone of a morphism from an object of $i\mathcal{A}$ to one of $G\mathcal{B}$.

Proposition 2.3.3. *There exists a natural transformation $K \xrightarrow{\alpha} Q$ fitting into a triangle*

$$KGE \xrightarrow{\gamma} K \xrightarrow{\alpha} Q \xrightarrow{\beta} KGE[1].$$

Proof. Apply the functor K to the triangle

$$GE \xrightarrow{\xi} \mathrm{id}_{\mathcal{C}} \xrightarrow{q} iQ \xrightarrow{\partial} GE[1];$$

and use that $KiQ \cong K$. □

We will use the natural transformation α in the definition of categorical deformation.

2.3.4 Gluing dg categories

We now recall some constructions which allow, given two dg categories and a quasi-functor between them, to construct an explicit recollement; for more details, see [KL12, Section 4.1], or [CDW24] for a treatment in terms of ∞-categories. Let $\mathcal{A}, \mathcal{B}$ be pretriangulated dg categories and ϕ a $\mathcal{B}$-$\mathcal{A}$ right quasi-representable bimodule. The category $\mathcal{A} \times_\phi \mathcal{B}$ has as objects the triples (M_1, M_2, n) with $M_1 \in \mathcal{A}$, $M_2 \in \mathcal{B}$ and $n \in \phi(M_2, M_1)$ a closed degree 0 element. The Hom-complexes are given by

$$\mathrm{Hom}_{\mathcal{A}\times_\phi\mathcal{B}}((M_1, M_2, n), (N_1, N_2, n')) = \mathrm{Hom}_{\mathcal{A}}(M_1, N_1) \oplus \mathrm{Hom}_{\mathcal{B}}(M_2, N_2) \oplus \phi[-1]$$

with differential

$$d(f_1, f_2, f_{21}) = (df_1, df_2, -df_{21} + n' f_1 - f_2 n).$$

The category $\mathcal{A} \times_\phi \mathcal{B}$ is pretriangulated and comes equipped with several functors:

- Two embeddings

$$\begin{aligned} i\colon \mathcal{A} &\to \mathcal{A} \times_\phi \mathcal{B} \\ M &\to (M, 0, 0) \end{aligned}$$

  and

$$\begin{aligned} G\colon \mathcal{B} &\to \mathcal{A} \times_\phi \mathcal{B} \\ M &\to (0, M, 0) \end{aligned}$$

- A left adjoint

$$\begin{aligned} Q\colon \mathcal{A} \times_\phi \mathcal{B} &\to \mathcal{A} \\ (M_1, M_2, n) &\to M_1 \end{aligned}$$

  to i and a right adjoint

$$\begin{aligned} E\colon \mathcal{A} \times_\phi \mathcal{B} &\to \mathcal{B} \\ (M_1, M_2, 0) &\to M_2 \end{aligned}$$

  to G;

- A right adjoint to i

$$\begin{aligned} K\colon \mathcal{A} \times_\phi \mathcal{B} &\to \mathrm{Mod}\,\mathcal{A} \\ (M_1, M_2, n) &\to \mathrm{Cone}(n)[-1]. \end{aligned}$$

  where we have used the Yoneda Lemma to see n as a map $h_{M_1} \to \phi(M_2, -)$.

The functors i, G, Q, E are honest dg functors while in principle K is only a bimodule; one can however show that it is always a quasi-functor. These functors satisfy the identities

$$Qi = Ki = \mathrm{id}_{\mathcal{A}}\,,\ EG = \mathrm{id}_{\mathcal{B}}\,,\ QG = 0,\ KG = \phi[-1],\ Ei = 0. \tag{2.7}$$

Note that the first two identities are induced by the (co)units of the respective adjunctions, expressing the fact the the (co)unit isomorphism is the identity.

Proposition 2.3.4. *The functors above fit into a recollement*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] & \mathcal{A} \times_\phi \mathcal{B} \arrow[l, bend right, "K"'] \arrow[l, bend left, "Q"] \arrow[r, "E"] & \mathcal{B}. \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

Proof. As already observed, the composition iE is the zero functor and the functors i and G are fully faithful. Hence we are only left with showing the existence of the required triangle. For this, observe that

$$GE(M_1, M_2, n) = (0, M_2, 0),\ iQ(M_1, M_2, n) = (M_1, 0, 0)$$

and the natural transformations $GE \xrightarrow{\xi} \mathrm{id}_{\mathcal{C}}$ and $\mathrm{id}_{\mathcal{C}} \xrightarrow{q} iQ$ are given by the obvious maps. There is a natural transformation $iQ \xrightarrow{\partial} GE[1]$ given by

$$\partial_{(M_1,M_2,n)} \colon (M_1, 0, 0) \xrightarrow{(0,0,n)} (0, M_1[1], 0)$$

which we claim fits into a triangle

$$GE \xrightarrow{\xi} \mathrm{id}_{\mathcal{A}\times_\phi\mathcal{B}} \xrightarrow{q} iQ \xrightarrow{\partial} GE[1].$$

To see this, one uses the description of cones in $\mathcal{A} \times_\phi \mathcal{B}$ from [KL12, Lemma 4.3]. Indeed, it is straightforward to check that the cone of $\partial[-1]$ is isomorphic the identity functor, and that the natural transformations ξ and q correspond to the canonical maps

$$GE \to \mathrm{Cone}(\partial[-1]) \to iQ.$$

$\square$

Like in the general case, we also obtain a (canonical) triangle

$$K \xrightarrow{\alpha} Q \xrightarrow{\beta} KGE[1] \xrightarrow{\gamma} K[1]$$

which, since $KG[1] = \phi$, is the same thing as a triangle

$$K \xrightarrow{\alpha} Q \xrightarrow{\beta} \phi E \xrightarrow{\gamma} K[1]. \tag{2.8}$$

Chapter 3

The n-derived category

This chapter contains the first significant results of the thesis. Given a curved deformation A_n of a dg-algebra A, we construct (Section 3.1) the n-derived category $D^n(A_n)$ as the quotient of the homotopy category of cdg A_n-modules by the modules which are filtered acyclic with respect to the t-adic filtration. We prove that this is a cofibrantly generated triangulated category (Section 3.2) which admits projective (Section 3.3) and injective (Section 3.3.2) resolutions; we also show (Section 3.4) that a bimodule X_n induces an equivalence of n-derived categories if and only if its reduction $X \otimes_{k[\varepsilon]} k$ is a Morita equivalence. The main result (Section 3.5) of this chapter is the fact that the n-derived category admits a semiorthogonal decomposition into $n+1$ copies of $D(A)$. In Section 3.6, we compare our construction with the ones from [Pos18], establishing Positselski's semiderived category as an admissible subcategory of $D^n(A_n)$; we also (Section 3.7) describe a model structure presenting the n-derived category. Finally in Section 3.8, we study the case of a formal deformation.

3.1 The n-derived category

In this section we define the n-derived category of a deformation, and prove some elementary results about them.

3.1.1 Construction of the n-derived category

Let A be a dg algebra over k and A_n a cdg deformation. Given a cdg A_n-module M, we can define both the t-adic filtration

$$0 = t^{n+1}M \subseteq t^n M \subseteq \ldots \subseteq tM \subseteq M$$

and the K-filtration

$$0 \subseteq \operatorname{Ker} t_M \subseteq \ldots \subseteq \operatorname{Ker} t_M^n \subseteq \operatorname{Ker} t_M^{n+1} = M.$$

Since $c \in tA_n$, we have that $d_M^2(t^iM) \subseteq t^{i+1}M$ so the successive quotients with respect to the t-adic filtration have induced predifferentials squaring to zero i.e. are actual complexes; since $t \operatorname{Ker} t_M^i \subseteq \operatorname{Ker} t_M^{i-1}$, the same is true for the K-filtration. We will write $\operatorname{Gr}_t(M)$ for the associated graded with respect to the t-adic filtration, and $\operatorname{Gr}_K(M)$ for that with respect to the K-filtration; by the above discussion $\mathrm{Gr}_t(M)$ and $\mathrm{Gr}_K(M)$ are complexes even when M is not. Moreover since t^i_M is a morphism of A_n-modules, t^iM and $\mathrm{Ker}\, t^i_M$ are themselves A_n-modules. Then, since t_M annihilates the graded pieces of either filtration, the A_n-module structure on those reduces to a natural A-module structure defining thus functors

$$\mathrm{Gr}_K(-), \mathrm{Gr}_t(-)\colon \ \mathrm{Mod}\, A_n \to \mathrm{Mod}\, A;$$

Our notion of acyclicity for curved modules will involve the acyclicity of these objects. Explicitly, $\mathrm{Gr}_t(M)$ is acyclic if the $n+1$ complexes

$$\mathrm{Gr}^i_t(M) = \frac{t^iM}{t^{i+1}M} \quad i \in 0, \dots, n$$

are acyclic; analogously, $\mathrm{Gr}_K(M)$ is acyclic if the $n+1$ complexes

$$\mathrm{Gr}^i_K(M) = \frac{\mathrm{Ker}\, t^{i+1}_M}{\mathrm{Ker}\, t^i_M} \cong t^i\, \mathrm{Ker}\, t^{i+1}_M \quad i \in 0, \dots, n$$

are acyclic; the last isomorphism follows from the obvious fact that, since $\mathrm{Ker}\, t^n_M \subseteq \mathrm{Ker}\, t^{n+1}_M$, the kernel of the restricted action of t^i_M on $\mathrm{Ker}\, t^{i+1}_M$ is still $\mathrm{Ker}\, t^i_M$.

Lemma 3.1.1. *For any cdg A_n-module M,*

$$t^iM \cap \mathrm{Ker}\, t^j_M = t^i\, \mathrm{Ker}\, t^{i+j}_M \subseteq M.$$

Proof. This is a straightforward elementwise verification.

□

The following is a key fact:

Proposition 3.1.2. *For any A_n-module M, the complex $\mathrm{Gr}_t(M)$ is acyclic if and only if the complex $\mathrm{Gr}_K(M)$ is acyclic.*

Proof. Suppose first that $\mathrm{Gr}_K(M)$ is acyclic. The n-th quotient of the K-filtration is

$$\frac{\mathrm{Ker}\, t^{n+1}_M}{\mathrm{Ker}\, t^n_M} = \frac{M}{\mathrm{Ker}\, t^n_M} \cong t^nM$$

which is thus acyclic. We then have the short exact sequence

$$0 \to \frac{\mathrm{Ker}\, t_M \cap t^{i-1}M}{\mathrm{Ker}\, t_M \cap t^iM} \to \frac{t^{i-1}M}{t^iM} \xrightarrow{t} \frac{t^iM}{t^{i+1}M} \to 0$$

which by Lemma 3.1.1 we can rewrite as

$$0 \to \frac{t^{i-1}\, \mathrm{Ker}\, t^i_M}{t^i\, \mathrm{Ker}\, t^{i+1}_M} \to \frac{t^{i-1}M}{t^iM} \xrightarrow{t} \frac{t^iM}{t^{i+1}M} \to 0. \tag{3.1}$$

Since $\mathrm{Gr}_K(M)$ is acyclic the first term is the quotient of two acyclic complexes, and by induction we get that each graded piece of the T-filtration is acyclic. For the opposite implication the argument is similar: assume that $\mathrm{Gr}_t(M)$ is acyclic. From the acyclicity of $\mathrm{Gr}_t(M)$ we know that $t^n M \cong \frac{M}{\operatorname{Ker} t_M^n}$ is acyclic; hence using again (3.1) and the short exact sequence

$$0 \to t^i \operatorname{Ker} t_M^{i+1} \to t^{i-1} \operatorname{Ker} t_M^i \to \frac{t^{i-1} \operatorname{Ker} t_M^i}{t^i \operatorname{Ker} t_M^{i+1}} \to 0$$

we conclude again by induction (now starting from the last piece) that $t^i \operatorname{Ker} t_M^{i+1}$ is acyclic for all i. □

Definition 3.1.3. An A_n-module is said to be *n-acyclic* if either of the two equivalent conditions of Proposition 3.1.2 is satisfied. A closed morphism $f\colon M \to N$ is said to be a *n-quasi-isomorphism* if its cone is n-acyclic.

Clearly, a closed morphism is a n-quasi-isomorphism if and only if it induces quasi-isomorphisms between the associated graded of either the t-adic or the K-filtration.

Example 3.1.1. If A_n is a dg algebra (i.e. $c = 0$), any n-acyclic module has by definition a finite filtration with acyclic quotients and is therefore itself acyclic. On the other hand, not all acyclic modules are n-acyclic. For an explicit example, consider the first order deformation $R_1 = k[t]/(t^2)$ of the base ring, and look at the R_1-module

$$M = \ldots \to R_1 \xrightarrow{t} R_1 \xrightarrow{t} R_1 \to \ldots$$

for an example with R_1-free components and its truncation

$$N = 0 \to k \xrightarrow{t} R_1 \to k \to 0$$

for a bounded example. The module M, however, is the only source of examples with R_1-free components. Indeed, it follows e.g. from [Low05, Theorem 5.7] (see also [Pos18, Section 0.15]) that any complex of R_1-free modules whose reduction is acyclic must already be contractible; now, the homotopy category of acyclic complexes of R_1-free modules is known to be a model for the (large) singularity category of R_1 (see e.g. [Kra05]), and is cofibrantly generated by M (as easily follows e.g. from [Jør05]). Hence, all examples of acyclic dg R_1-modules with R_1-free components are obtained (up to homotopy equivalence) from M via shifts, cones and coproducts.

As soon as non-free components are let back into the pictures, the class of examples expands dramatically.

Corollary 3.1.4. *If M is n-acyclic, then $\operatorname{Ker} t_M^i$ is $(i-1)$-acyclic for all $i > 0$.*

Proof. The graded pieces of $\operatorname{Ker} t_M^i$ with respect to the K-filtration are a subset of the graded pieces of M with respect to the same filtration, so this is implied by Proposition 3.1.2. □

Remark. At a first glance, our constructions have a lot in common with the theory of N-complexes [Kap96]; indeed if A_n is a deformation of order n, since $d_M^{2n+2}(M) \subseteq t^{n+1}M = 0$ any A_n-module is a $(2N + 2)$-complex. However the two notions of acyclicity differ significantly. Consider the case of a first order deformation A_1 with zero curvature, and take an A_1-module M; looking at it as a 4-complex, being 4-exact in the sense of [Kap96] implies that the module

$$ {}_2H^i(M) = \frac{\operatorname{Ker} d_M^2 \colon M^i \to M^{i+2}}{\operatorname{Im} d_M^2 \colon M^{i-2} \to M^i} $$

vanishes. But since A_1 has zero curvature, $d_M^2 = 0$ and ${}_2H^i(M) = M^i$; therefore any 4-exact A_1-module has to be the zero module. On the other hand any contractible module suffices in giving an example of a nonzero A_1 module which is 1-acyclic according to our definition.

Proposition 3.1.5. *The full subcategory* $\mathrm{Ac}^n \subseteq \mathrm{Hot}(A_n)$ *given by the n-acyclic modules is a triangulated subcategory closed under small products and coproducts.*

Proof. We first verify that Ac^n is closed under isomorphisms in $\mathrm{Hot}(A_n)$, i.e. homotopy equivalences: this follows from the fact that if $f\colon M \to N$ is a homotopy equivalence, it induces homotopy equivalences $\mathrm{Gr}_t^i(f)\colon \mathrm{Gr}_t^i(M) \to \mathrm{Gr}_t^i(N)$. Similarly, if M and N are n-acyclic and $f\colon M \to N$ is any closed morphism, we have isomorphisms

$$ \mathrm{Gr}_t^i(\mathrm{Cone}(M \xrightarrow{f} N)) \cong \mathrm{Cone}(\mathrm{Gr}_t^i(M) \xrightarrow{\mathrm{Gr}_t^i(f)} \mathrm{Gr}_t^i(N)) $$

and $\mathrm{Cone}(f)$ is n-acyclic.

We have then proved that $\mathrm{Ac}^n \subseteq \mathrm{Hot}(A_n)$ is a triangulated subcategory. To show that it is closed under products and coproducts it is enough to show, for example, that $\mathrm{Gr}_t^i(-)$ commutes with products and coproducts. For this, observe that $\mathrm{Gr}_t^i(-)$ coincides with the composition

$$ \mathrm{Mod}\, A_n \xrightarrow{\mathrm{Im}\, t^i} \mathrm{Mod}\, A_{n-i} \xrightarrow{A \otimes_{A_{n-i}} -} \mathrm{Mod}\, A; $$

the functor $\mathrm{Im}\, t^i$ commutes with both products and coproducts; the tensor functor is a right adjoint so it automatically commutes with coproducts and, since A is finitely-presented as an A_{n-i}-module, also commutes with products [Sta, Tag 059K][1] . ☐

Remark. In fact, it is immediate to see that $\mathrm{Gr}_K^i(-)$ also commutes with (co)products, so both reductions preserve products and coproducts.

We now arrive at our main definition.

Definition 3.1.6. The *n-derived category* $D^n(A_n)$ is the quotient of $\mathrm{Hot}(A_n)$ by the subcategory $\mathrm{Ac}^n \subseteq \mathrm{Hot}(A_n)$.

In the case $n = 0$, the 0-acyclic modules coincide with the acyclic ones and $D^0(A) = D(A)$.

By Proposition 3.1.5, $D^n(A_n)$ is a triangulated category with small products and coproducts and the quotient functor

$$ \mathrm{Hot}(A_n) \to D(A_n) $$

preserves them.

We now give the relevant version of the notions of homotopy projective and homotopy injective modules.

[1] The reference is for commutative rings, but in our case it is sufficient since $A \otimes_{A_{n-i}} M \cong k \otimes_{R_{n-i}} M$ as k-modules.

Definition 3.1.7. We will say that an A_n-module P is *n-homotopy projective* if

$$\mathrm{Hom}_{\mathrm{Hot}(A_n)}(P, N) = 0$$

for any n-acyclic module N. An A_n-module I is *n-homotopy injective* if

$$\mathrm{Hom}_{\mathrm{Hot}(A_n)}(N, I) = 0$$

for any n-acyclic module N.

In analogy to the classical case, n-homotopy projective modules are those in ${}^{\perp}\mathrm{Ac}^n$ and n-homotopy injective modules those in $\mathrm{Ac}^{n\perp}$.

3.1.2 First results

Proposition 3.1.8. *The module $A \in \mathrm{Mod}\, A_n$ is n-homotopy projective.*

Proof. If M is an n-acyclic A_n-module, we have

$$\mathrm{Hom}_{A_n}(A, M) \cong \mathrm{Hom}_A(A, \mathrm{Ker}\, t_M) \cong \mathrm{Ker}\, t_M$$

which is acyclic by Lemma 3.1.2. □

We can now show the first way in which $D^n(A_n)$ differs significantly from other derived categories of curved objects in the literature. Observe preliminarily that the restriction functor $\mathrm{Hot}(A) \to \mathrm{Hot}(A_n)$ carries acyclic modules to n-acyclic modules, so defines a functor $D(A) \to D^n(A_n)$.

Corollary 3.1.9. *The natural functor $D(A) \to D^n(A_n)$ is fully faithful.*

Proof. Since A is n-homotopy projective as an A_n-module, we have isomorphisms

$$\mathrm{Hom}_{D(A)}(A, M) \cong \mathrm{Hom}_{\mathrm{Hot}(A)}(A, M) \cong \mathrm{Hom}_{\mathrm{Hot}(A_n)}(A, M) \cong \mathrm{Hom}_{D^n(A_n)}(A, M).$$

for all A-modules M. Since A is a compact generator of $D(A)$, by [Kel94, Lemma 4.2] this implies the claim. □

Corollary 3.1.9 will have a significant generalization in Corollary 3.3.13.

Corollary 3.1.10. *If A_n is a dg algebra, there is a fully faithful functor*

$$D(A_n) \hookrightarrow D^n(A_n).$$

Proof. Recall that the quotient $\mathrm{Hot}(A_n) \to D(A_n)$ admits a fully faithful left adjoint $\mathbf{p}\colon D(A_n) \to \mathrm{Hot}(A_n)$ whose essential image is given by the homotopy projective A_n-modules; composing this with the quotient $\mathrm{Hot}(A_n) \to D(A_n)$ one obtains a functor

$$D(A_n) \xrightarrow{\mathbf{p}} \mathrm{Hot}(A_n) \to D^n(A_n).$$

Since every n-acyclic module is acyclic, homotopy projective modules are in particular n-homotopy projective, so the quotient $\mathrm{Hot}(A_n) \to D^n(A_n)$ is fully faithful when restricted to the image of $\mathbf{p}$ and we are done. □

Remark. In other words, the quotient functor

$$D^n(A_n) \to D(A_n)$$

admits both a left and a right adjoint, which are then automatically fully faithful.

3.2 Compact generation

So far, we haven't proven anything about $D^n(A_n)$ which is not also true about $\mathrm{Hot}(A_n)$; in particular, $D^n(A_n)$ might be too large for it to represent a useful invariant. In this section we prove that that is not the case by showing that, unlike $\mathrm{Hot}(A_n)$, the n-derived category is a cofibrantly generated triangulated category (Theorem 3.2.5).

Example 3.2.1. If A_n is a dg algebra, it is easy to see that $D^n(A_n)$ is compactly generated. Indeed consider the $n+1$ dg A_n-modules $A_0, A_1, \ldots, A_n$. Those are all n-homotopy projective, since for any n-acyclic module M one has

$$\mathrm{Hom}_{A_n}(A_i, M) \cong \mathrm{Hom}_{A_i}(A_i, \mathrm{Ker}\, t_M^{i+1}) \cong \mathrm{Ker}\, t_M^{i+1}$$

which is i-acyclic and hence acyclic.

Now if M is an A_n-module such that $\mathrm{Ker}\, t_M^{i+1}$ is acyclic for all i, it follows immediately that $\mathrm{Gr}_K(M)$ is acyclic and therefore M is n-acyclic; compactness comes from the fact that the functors $\mathrm{Ker}\, t^i$ commute with coproducts.

The problem is that if A_n has nonzero curvature, none of the modules A_i - except A_0 - exist in $\mathrm{Mod}\, A_n$. The question is then one of finding alternative compact generators that are defined also in the curved case.

Remark. Assume again that A_n has no curvature. Then one can show that n-acyclic modules coincide with the modules which are contractible over $k[t]/(t^{n+1})$; indeed, it follows from [Jør05] that any A_n-module M for which $\mathrm{Ker}\, t_M^i$ is acyclic for all i is the zero object in $\mathrm{Hot}(k[t]/(t^{n+1}))$ – i.e. is contractible over $k[t]/(t^{n+1})$. Hence, the category $D^n(A_n)$ coincides with the so called *relative derived category* of A_n. It was shown in [Nic08] that this is equivalent to the Bar derived category of A_n – that is, the category of A_∞ A_n-modules up to A_∞ homotopy. Note that this does *not* coincide with the usual derived category, since $k[t]/(t^{n+1})$ is not a field. We expect that in general there exists an equivalence between the n-derived category and the Bar derived category studied in [Nic08].

3.2.1 Twisted modules

Definition 3.2.1. A right (resp. left) *qdg module* [PP12; DL18] over a cdg algebra $\mathcal{A}$ is a graded right (resp. left) $\mathcal{A}^\#$-module $M^\#$ equipped with a derivation $d_M \in \mathrm{Hom}_R(M^\#, M^\#)$ of degree 1.

It is evident from this definition that a right cdg module is a right qdg module for which the condition $d_M^2 = -c$ holds. The key reason why qdg modules are relevant is that, while the free module $\mathcal{A}$ is not a cdg module, it is a qdg module. Although for us this notion is just a technical tool to eventually get back to the world of cdg modules, it is one with intrinsic interest: in the many-objects case, considering the larger category of qdg modules makes it possible to define a Yoneda embedding, which does not exist in the cdg case ([DL18; PP12]).

Let now M_i be a a finite family of qdg A_n-modules and

$$F\colon \bigoplus_i M_i \to \bigoplus_i M_i$$

an A_n-linear morphism of degree 1. The twisted module M_F is defined as the qdg module which has as underlying graded module $\oplus_i M_i$ and as predifferential

$$d_F m_i = d_{M_i} m_i + F m_i.$$

It is straightforward to see that d_F is still a derivation, so M_F is in a natural way a qdg module; we will say that M_F is the qdg module $\oplus_i M_i$ twisted by F. The condition $d_F^2 m_i = c m_i$ of being a cdg module corresponds to the Maurer-Cartan equation for F

$$d_{M_i}^2 m_i + F d_{M_i} + d_{M_i} F + F^2 m_i = c m_i.$$

Remark. In the A_∞-case, infinite sums get involved and delicate convergence issues arise, see [LVdB15]; in the dg case that we placed ourselves in, any morphism F gives a well defined twisted module.

3.2.2 Construction of the generators

To define the modules that will be the generators of $D^n(A_n)$, begin with the following observations: we know that A_n, as well as A_i for $i \leq n$, is a qdg A_n-module; moreover, there is a well-defined A_n-module map

$$t\colon A_{i-1} \to A_i$$

for all i, together with the already cited projection $\pi\colon A_i \to A_{i-1}$. Finally since $c \in tA_n$, there exists an element $\frac{c}{t} \in A_n^2$ such that $t\frac{c}{t} = c$.

Set $A_{-1} = 0$. Define the twisted module Γ_i for $i = 0, \dots, n$ as the qdg module $X_i = A_i \oplus A_{i-1}[1]$ twisted by the matrix

$$\gamma_i = \begin{bmatrix} 0 & \pi \circ \frac{c}{t} \\ -t & 0 \end{bmatrix}.$$

Concretely, one has

$$d_{\Gamma_i}(\alpha_k, a_{k+1}) = (d_{A_n}\alpha_k - t a_{k+1}, d_{A_{n-1}[1]} a_{k+1} + \pi\left(\frac{c}{t}\alpha_k\right))$$

for $\alpha_k \in A_i$ and $a_{k+1} \in A_{i-1}$; recall that by definition $d_{A_{i-1}[1]} = -d_{A_{i-1}}$.

Proposition 3.2.2. *The qdg A_n-module Γ_n is a cdg A_n-module.*

Proof. For simplicity of notation, write γ for γ_n and X for X_n. We have to verify that, for all $x \in X$,

$$d_{\Gamma_n}^2 x = d_X^2 x + d_X \gamma x + \gamma d_X + \gamma^2 x = cx.$$

Let's examine the sum part by part.

- Since $\pi(c)$ coincides with the curvature of A_{n-1}, we have

$$d_X^2(\alpha_k, a_{k+1}) = (c\alpha_k - \alpha_k c, \pi(c)a_{k+1} - a_{k+1}\pi(c)).$$

- For the last term, we have

$$\gamma^2(\alpha_k, a_{k+1}) = (-t\pi\left(\frac{c}{t}\alpha_k\right), -\pi\left(t\frac{c}{t}a_{k+1}\right)) = (-c\alpha_k, -\pi(c)a_{k+1})$$

  where the last equality is given by direct computation.

Therefore $d_X^2 + \gamma^2$ equals the action of the curvature, and we will be done if we show that

$$d_X\gamma + \gamma d_X = 0.$$

Computing, we get

$$d_X\gamma(\alpha_k, a_{k+1}) = (-d_{A_n} t a_{k+1}, -d_{A_{n-1}}\pi\left(\alpha_k \frac{c}{t}\right))$$

and

$$\gamma d_X(\alpha_k, a_{k+1}) = (t d_{A_{n-1}} a_{k+1}, \pi\left(d_{A_n}\alpha_k \frac{c}{t}\right));$$

since t commutes with the predifferentials, the first terms cancel out; moreover π commutes with the predifferentials so the sum of the second terms reduces to

$$\pi\left(\alpha_k d_{A_n}\frac{c}{t}\right) = \pi(\alpha_k)\pi\left(d_{A_n}\frac{c}{t}\right)$$

and in order to conclude we need to prove that $\pi\left(d_{A_n}\frac{c}{t}\right) = 0$. Since c is closed with respect to d_{A_n}, one has

$$0 = d_{A_n}c = t d_{A_n}\frac{c}{t}$$

so $d_{A_n}\frac{c}{t} \in \operatorname{Ker} t_{A_n}$. But A_n is R_n-free, so $\operatorname{Ker} t_{A_n} = t^n A_n = \operatorname{Ker}\pi$ and we are done. $\square$

Remark. The module Γ_n is similar to the module $A_{(n)}$ defined in [LVdB15] as the twist of $A_n \oplus A_n[1]$ by the matrix

$$\begin{bmatrix} 0 & \frac{c}{t} \\ t & 0 \end{bmatrix}.$$

However in order for $A_{(n)}$ to be a cdg module, the authors need to impose the existence of a deformation A_{n+1} of a higher order extending the deformation A_n; this makes it possible to "correct" $\frac{c}{t}$ into an actual d_{A_n}-closed element. In our case, as we have shown, this need is removed by quotienting out $t^n A_n[1]$. Of course, the module Γ_n does not work for the scope of [LVdB15] since it is not R_n-free.

In the same way, one proves that Γ_i is a cdg A_n-module for all i. For a cdg A_n-module M, denote by $(M)_i$ the complex of R_n-modules $\mathrm{Hom}_{A_n}(\Gamma_i, M)[1]$; explicitly[2], $(M)_i$ is the module $\operatorname{Ker} t_M^i \oplus \operatorname{Ker} t_M^{i+1}[1]$ twisted by the matrix

$$\begin{bmatrix} 0 & \iota \circ \frac{c}{t} \\ t & 0 \end{bmatrix} \tag{3.2}$$

where $\iota\colon \operatorname{Ker} t_M^i \to \operatorname{Ker} t_M^{i+1}$ is the inclusion; note that $t \operatorname{Ker} t_M^{i+1} \subseteq \operatorname{Ker} t_M^i$ so everything is well defined. By definition $\Gamma_0 = A$ so $(M)_0 \cong \operatorname{Ker} t_M[1]$.

Proposition 3.2.3. *There is a triangle in $D(R_n)$*

$$\operatorname{Ker} t_M[1] \to (M)_n \to \frac{\operatorname{Ker} t_M^n}{tM} \to \operatorname{Ker} t_M[2]. \tag{3.3}$$

Proof. There are natural chain maps

$$\operatorname{Ker} t_M[1] \to (M)_n \text{ and } (M)_n \to \frac{\operatorname{Ker} t_M^n}{tM}$$

induced by the inclusion $\operatorname{Ker} t_M \hookrightarrow M$ and the projection $\operatorname{Ker} t_M^n \to \operatorname{Ker} t_M^n/tM$. These give short exact sequences

$$0 \to X \to (M)_n \to \frac{\operatorname{Ker} t_M^n}{tM} \to 0 \tag{3.4}$$

and

$$0 \to \operatorname{Ker} t_M[1] \to (M)_n \to Y \to 0 \tag{3.5}$$

where X is the twist of $tM \oplus M[1]$ and Y of $\operatorname{Ker} t_M^n \oplus M/\operatorname{Ker} t_M[1]$ by the same matrix (3.2) that defines $(M)_n$. One also sees that there are chain maps $\operatorname{Ker} t_M[1] \to X$ and $Y \to \operatorname{Ker} t_M^n/tM$ - again induced by the inclusion and projection - giving the short exact sequences

$$0 \to \operatorname{Ker} t_M[1] \to X \to Z \to 0$$

and

$$0 \to Z' \to Y \to \frac{\operatorname{Ker} t_M^n}{tM} \to 0.$$

It's immediate to see that Z and Z' are isomorphic, both being identified to the module $tM \oplus M/\operatorname{Ker} t_M[1]$ twisted by the matrix with the same coefficients as (3.2). Under the isomorphism $M/\operatorname{Ker} t_M \cong tM$, one sees that Z corresponds to the module $tM \oplus tM[1]$ twisted by the matrix

$$\begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix}.$$

Consider the t-adic filtration on Z; it is a finite filtration, and since c is divisible by t its graded pieces are the twists of $\frac{t^iM}{t^{i+1}M} \oplus \frac{t^iM}{t^{i+1}M}[1]$ by the matrix

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

this, being the cone of an isomorphism, is contractible: then each graded piece is acyclic, and so is Z. Thus we get that X is quasi-isomorphic to $\operatorname{Ker} t_M[1]$ and Y to $\operatorname{Ker} t_M^n/tM$. Plugging these identifications in either (3.4) or (3.5) we are done. $\square$

[2] The shift is just to avoid negative indices.

In the same way, one shows that there are triangles in $D(R_i)$

$$\operatorname{Ker} t_M[1] \to (M)_i \to \frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^{i+1}} \to \operatorname{Ker} t_M[2] \tag{3.6}$$

for all $i \leq n$.

Proposition 3.2.4. *The modules $\Gamma_0, \ldots, \Gamma_n \in \operatorname{Mod} A_n$ are n-homotopy projective compact A_n-modules.*

Proof. Let M be an n-acyclic A_n-module: by Proposition 3.1.2, we know that $\operatorname{Ker} t_M[1] \cong (M)_0$ is acyclic. Hence by the triangle (3.6) it is enough to prove that $\frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^{i+1}}$ is acyclic to conclude that $(M)_i$ is acyclic. For this, consider the short exact sequence

$$0 \to \frac{t \operatorname{Ker} t_M^{i+1}}{t \operatorname{Ker} t_M^i} \to \frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^i} \to \frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^{i+1}} \to 0$$

stemming from the third isomorphism theorem for modules. Applying Lemma 3.1.1 to $\operatorname{Ker} t_M^{i+1}$, we have that

$$t \operatorname{Ker} t_M^i = t \operatorname{Ker} t_M^{i+1} \cap \operatorname{Ker} t_M^{i-1}$$

so the first term can be rewritten as

$$\frac{t \operatorname{Ker} t_M^{i+1}}{t \operatorname{Ker} t_M^{i+1}} \cong \frac{t \operatorname{Ker} t_M^{i+1}}{t \operatorname{Ker} t_M^{i+1} \cap \operatorname{Ker} t_M^{i-1}} \cong t^i \operatorname{Ker} t_M^{i+1}$$

and the whole short exact sequence becomes

$$0 \to t^i \operatorname{Ker} t_M^{i+1} \to \frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^i} \to \frac{\operatorname{Ker} t_M^i}{t \operatorname{Ker} t_M^{i+1}} \to 0; \tag{3.7}$$

by Corollary 3.1.4 $\operatorname{Ker} t_M^{i+1}$ is i-acyclic and $\operatorname{Ker} t_M^i$ is $(i-1)$ acyclic, thus the first and second term are acyclic and hence so is the third: we have then proved that $(M)_i$ is acyclic for all n-acyclic M and all i, i.e. $\Gamma_0, \ldots, \Gamma_n$ are n-homotopy projective. Compactness comes directly from the fact that for any collection $\{M_\lambda\}_{\lambda \in \Lambda}$ there is an isomorphism

$$\bigoplus_\lambda (M_\lambda)_i \cong \left(\bigoplus_\lambda M_\lambda \right)_i$$

following from the fact that the functor $\operatorname{Ker} t^i$ commutes with coproducts. $\square$

Theorem 3.2.5. *The category $D^n(A_n)$ is compactly generated by the objects $\Gamma_0, \ldots \Gamma_n$.*

Proof. By induction on n, the case $n = 0$ being the usual statement that $A \in D(A)$ is a generator. Let M be a module such that $\operatorname{Hom}_{D^n(A_n)}(\Gamma_i, M[l]) = 0$ for all i and $l \in \mathbb{Z}$. Since the modules Γ_i are n-homotopy projective, this implies that $\operatorname{Hom}_{A_n}(\Gamma_i, M)$ is acyclic for all i. Since the modules $\Gamma_0, \ldots, \Gamma_{n-1}$ all lie in the image of the restriction functor from $\operatorname{Mod} A_{n-1}$ and are $(n-1)$-homotopy projective as A_{n-1}-modules, we know that

$$0 = \operatorname{Hom}_{\operatorname{Hot}(A_n)}(\Gamma_i, M[l]) \cong \operatorname{Hom}_{\operatorname{Hot}(A_{n-1})}(\Gamma_i, \operatorname{Ker} t_M^n[l]) \cong \operatorname{Hom}_{D^{n-1}(A_{n-1})}(\Gamma_i, \operatorname{Ker} t_M^n[l])$$

for all l and $i < n$, so $\operatorname{Ker} t_M^n$ is $(n-1)$-acyclic by the inductive hypothesis; in particular, $\operatorname{Ker} t_M$ is acyclic. Then, since $(M)_n$ and $\operatorname{Ker} t_M[1]$ are acyclic, the triangle (3.3) guarantees that $\frac{\operatorname{Ker} t_M^n}{tM}$ is acyclic and by the short exact sequence (3.7) for the case $i = n$ this implies that $t^n M$ is acyclic; finally, the $(n-1)$-acyclicity of $\operatorname{Ker} t_M^n$ together with the acyclicity of $t^n M$ means that $\operatorname{Gr}_K(M)$ is acyclic and therefore M is n-acyclic. □

Example 3.2.2. Let $k[u, u^{-1}]$ be the graded field of [KL09] with u in degree 2; let $k_u[u, u^{-1}]$ be the deformation over R_1 corresponding to the Hochschild cocycle $u \in k[u, u^{-1}]^2$. The category $D^1(k_u[u, u^{-1}])$ has as generators the two $k_u[u, u^{-1}]$-modules

$$\Gamma_0 = \ldots \to 0 \to k \to 0 \to k \to 0 \to k \to \ldots$$

and

$$\Gamma_1 = \ldots \to k \xrightarrow{t} R_1 \xrightarrow{t=0} k \xrightarrow{t} R_1 \to \ldots$$

It turns out - and this is a peculiarity of the deformation $k_u[u, u^{-1}]$ - that these generators are actually *orthogonal* to each other, i.e. there is no nonzero morphism in either direction. The endomorphism ring of Γ_0 is isomorphic to the base $k[u, u^{-1}]$ and the one of Γ_1 is quasi-isomorphic to the same algebra. In Section 3.5 we will explain this kind of behaviour in greater generality.

3.3 Resolutions

Using the fact that $D^n(A_n)$ has a set of n-homotopy projective compact generators we can already deduce formally the existence of n-homotopy projective resolutions. That said, we will still give a direct construction (Corollary 3.3.9) which gives some insight into the explicit shape of these resolutions. This explicit construction also adapts to give a proof of the existence of n-homotopy injective resolutions (Proposition 3.3.15).

3.3.1 Projective resolutions

Let $\mathcal{X} \subseteq \operatorname{Mod} A_n$ be a set of A_n-modules which are compact as objects of $\operatorname{Hot}(A_n)$.

Definition 3.3.1. An A_n-module P is $\mathcal{X}$*-semifree* if there exists a filtration

$$0 = F_0 P \subseteq F_1 P \subseteq \ldots \subseteq P$$

such that:

- $P = \cup_i F_i P$;
- The inclusions $F_i P \hookrightarrow F_{i+1} P$ split as morphisms of graded modules;
- $F_{i+1} P / F_i P$ is isomorphic to a direct sum of copies of shifts of elements of $\mathcal{X}$.

We will say that an A_n-module is *n-semifree* if it is $\{\Gamma_0, \ldots \Gamma_n\}$-semifree.

Definition 3.3.2. A module $M \in \operatorname{Hot}(A_n)$ is said to be $\mathcal{X}$*-cell* if it lies in the minimal localizing subcategory of $\operatorname{Hot}(A_n)$ containing $\mathcal{X}$.

We will say that an A_n-module M is *n-cell* if it is $\{\Gamma_0, \dots \Gamma_n\}$-cell. A module $N \in \mathrm{Hot}(A)$ is *A-cell* if it lies in the minimal localizing subcategory of $\mathrm{Hot}(A)$ containing the A-module A.

Lemma 3.3.3. *Any $\mathcal{X}$-semifree module is $\mathcal{X}$-cell.*

Proof. Same proof as [Sta, 09KL], *mutatis mutandis*: assume that P is an $\mathcal{X}$-semifree A_n-module. It is easy to see by induction that each F_iP is $\mathcal{X}$-cell, and to see that P is itself $\mathcal{X}$-cell one uses the existence of the graded split short exact sequence

$$0 \to \bigoplus_i F_iP \to \bigoplus_i F_iP \to P \to 0.$$

defined as in [Sta, 09KL].

□

Remark. If an A_n-module admits a filtration as in the definition of $\mathcal{X}$-semifree with the exception of the quotients being $\mathcal{X}$-cell instead of a coproduct of shifted copies of elements of $\mathcal{X}$, one can use the same argument to prove that it is itself $\mathcal{X}$-cell.

Proposition 3.3.4. *All n-cell A_n-modules are n-homotopy projective.*

Proof. The claim follows from the fact that $\Gamma_0, \dots, \Gamma_n$ are n-homotopy projective and that a coproduct of n-homotopy projectives is still n-homotopy projective. □

In particular, for the case $n = 0$ we recover the classical fact that all A-cell modules are homotopy projective.

Remark. It is well known that the property of being A-cell coincides with that of being homotopy projective as an A-module. By the end of this section, we will see that this generalizes to the filtered setting.

Lemma 3.3.5. *If $M \in \mathrm{Hot}(A_n)$ is n-cell, then both $\mathrm{Gr}_t(M)$ and $\mathrm{Gr}_K(M)$ are A-cell and therefore homotopy projective.*

Proof. It is immediate to see that the subcategory of $\mathrm{Hot}(A_n)$ given by the modules with A-cell associated graded is a triangulated subcategory containing $\Gamma_0, \dots \Gamma_n$. To conclude we need to show that it is also closed under coproducts, but this follows from the fact that that $\mathrm{Gr}_t(-)$ and $\mathrm{Gr}_K(-)$ commute with coproducts. □

If M is an A_n-module, an n-cell (resp. an n-semifree) resolution of M is an n-cell (resp. an n-semifree) A_n-module P equipped with a n-quasi-isomorphism $P \to M$.

Lemma 3.3.6. *If $P \to M$ is an n-cell resolution of $M \in \mathrm{Mod}\, A_n$, then*

$$\mathrm{Gr}_t(P) \to \mathrm{Gr}_t(M)$$

is a homotopy projective resolution of $\mathrm{Gr}_t(M)$ in $\mathrm{Hot}(A)$; the same holds for $\mathrm{Gr}_K(-)$.

Proof. By construction $\mathrm{Gr}_t(P) \to \mathrm{Gr}_t(M)$ is a quasi-isomorphism; moreover by Lemma 3.3.5, $\mathrm{Gr}_t(P)$ is a homotopy projective A-module. The case of $\mathrm{Gr}_K(-)$ is identical. □

The construction of n-cell resolutions uses the following procedure which is a generalization of the classical homotopy projective resolution of a dg module due to Keller [Kel94][Sta, 09KK].

Construction of projective resolutions

Lemma 3.3.7. *For every A_n-module M, there exists an $\mathcal{X}$-semifree module Q_X with a morphism $Q_X \to M$ inducing surjections*

$$\mathrm{Hom}_{A_n}(X, Q_X) \to \mathrm{Hom}_{A_n}(X, M).$$

and

$$\mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,Q_X)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,M)}.$$

for all $X \in \mathcal{X}$.

Proof. Pick an arbitrary $X \in \mathcal{X}$. By definition, $(\mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,M)})^0 = Z^0 \mathrm{Hom}_{A_n}(X, M)$. Choose a set $\{f^p\}$ of generators of $Z^0 \mathrm{Hom}_{A_n}(X, M)$ as an abelian group; each f^p defines tautologically a closed morphism

$$f^p \colon X \to M$$

with the property that f^p lies in the image of

$$f^p_* \colon Z^0 \mathrm{Hom}_{A_n}(X, X) \to Z^0 \mathrm{Hom}_{A_n}(X, M)$$

as the pushforward of the identity. Defining $Q'_X = \oplus_p X$, the collection $\{f^p\}$ defines a closed morphism $F \colon Q'_X \to M$ such that the precomposition with the inclusion in the p-th factor $X \to \bigoplus_p X$ equals f^p; it follows that F induces a surjection

$$Z^0 \mathrm{Hom}_{A_n}(X, Q_X) \to Z^0 \mathrm{Hom}_{A_n}(X, M).$$

To get surjectivity at every degree, repeat the same procedure adding appropriate shifts of X to Q'_X. For the surjectivity on arbitrary morphisms one proceeds similarly: let C_X be the cocone of the identity of X. Let $\{g^q\}$ be a set of (not necessarily closed) generators of $\mathrm{Hom}_{A_n}(X, M)^0$ as an abelian group. The pairs (g^q, dg^q) define closed morphisms $\gamma^q \colon C_X \to M$, and there is a natural non-closed morphism $\kappa \colon X \to C_X$ with the property that the composition with the inclusion into the q-th factor $X \xrightarrow{\iota} C_X \xrightarrow{\gamma^q} M$ equals g^q; therefore g^q lies in the image of the composition

$$\mathrm{Hom}_{A_n}(X, X)^0 \xrightarrow{\kappa_*} \mathrm{Hom}_{A_n}(X, C_X)^0 \xrightarrow{\gamma^j_*} \mathrm{Hom}_{A_n}(X, M)^0$$

as the image of the identity of X. Setting again $Q''_X = \oplus_q C_X$, we have an induced map $G \colon Q''_X \to M$ which gives rise to a surjection

$$\mathrm{Hom}_{A_n}(X, Q''_X)^0 \to \mathrm{Hom}_{A_n}(X, M)^0;$$

as before, repeat the whole procedure with shifts of C_X to obtain surjectivity at all degrees. To get the desired morphism, just set

$$Q_X = \bigoplus_{X \in \mathcal{X}} Q'_X \oplus Q''_X$$

and consider the induced map to M; Q_X has a 2-step filtration as in the definition of $\mathcal{X}$-semifree so it is $\mathcal{X}$-semifree and for any $X \in \mathcal{X}$, the morphism

$$\mathrm{Hom}_{A_n}(X, Q_X) \to \mathrm{Hom}_{A_n}(X, M)$$

is both surjective and surjective when restricted to the cycles.

□

Proposition 3.3.8. *For any A_n-module M there exists an $\mathcal{X}$-semifree module P_X with a closed morphism*

$$P_X \to M$$

inducing a quasi-isomorphism

$$\mathrm{Hom}_{A_n}(X, P_X) \xrightarrow{\sim} \mathrm{Hom}_{A_n}(X, M).$$

for all $X \in \mathcal{X}$.

Remark. If A_n is a dg algebra and by choosing $\mathcal{X} = \{A_n\}$, one obtains the same resolution as in [Kel94].

Proof. Apply Lemma 3.3.7 to obtain a closed morphism $P_0 \to M$, and denote by K_0 its kernel: since both $\mathrm{Hom}_{A_n}(X, -)$ and $\mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,-)}$ are left exact functors, by construction both the sequences

$$0 \to \mathrm{Hom}_{A_n}(X, K_0) \to \mathrm{Hom}_{A_n}(X, P_0) \to \mathrm{Hom}_{A_n}(X, M) \to 0$$

and

$$0 \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,K_0)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,P_0)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,M)} \to 0$$

are exact. Applying again Lemma 3.3.7 to obtain a morphism $P_1 \to K_0$, we can iterate the procedure to obtain a sequence

$$\ldots \to P_2 \to P_1 \to P_0 \to M$$

such that

$$\ldots \to \mathrm{Hom}_{A_n}(X, P_2) \to \mathrm{Hom}_{A_n}(X, P_1) \to \mathrm{Hom}_{A_n}(X, P_0) \to \mathrm{Hom}_{A_n}(X, M) \to 0$$

and

$$\ldots \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,P_2)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,P_1)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,P_0)} \to \mathrm{Ker}\, d_{\mathrm{Hom}_{A_n}(X,M)} \to 0$$

are exact for all $X \in \mathcal{X}$. Set $P_X = \mathrm{Tot}^{\oplus}(\ldots \to P_2 \to P_1 \to P_0)$; the augmentation $P_0 \to M$ induces a closed morphism $P_X \xrightarrow{\pi} M$. Since X is compact, the cone of π_* is isomorphic to

$$\mathrm{Tot}^{\oplus}(\ldots \to \mathrm{Hom}_{A_n}(X, P_2) \to \mathrm{Hom}_{A_n}(X, P_1) \to \mathrm{Hom}_{A_n}(X, P_0) \to \mathrm{Hom}_{A_n}(X, M) \to 0)$$

which by [Sta, 09IZ] is exact so

$$\pi_* \colon \mathrm{Hom}_{A_n}(X, P_X) \to \mathrm{Hom}_{A_n}(X, M)$$

is a quasi-isomorphism for all X.

To conclude that P_X is $\mathcal{X}$-semifree, recalling that each P_i has a 2-step filtration $0 \hookrightarrow S \hookrightarrow P$ as in Proposition 3.3.3 we can define a filtration F_i on P_X as

$$F_{2i}P_X = \mathrm{Tot}(0 \to P_i \to \ldots \to P_0 \to 0)$$

and adding S_{i+1} to obtain $F_{2i+1}P_X$. It is then readily seen that the filtration F_i satisfies the conditions in the definition of $\mathcal{X}$-semifree module.

□

Corollary 3.3.9. *Any A_n-module M admits an n-semifree, and hence n-cell, resolution.*

Proof. Apply Proposition 3.3.8 with $\mathcal{X} = \{\Gamma_0, \ldots \Gamma_n\}$ to obtain an n-semifree module P with a morphism $P \to M$; by construction

$$\mathrm{Hom}_{A_n}(\Gamma_i, P) \to \mathrm{Hom}_{A_n}(\Gamma_i, M)$$

is a quasi-isomorphism for all i. Since the modules Γ_i are generators of $D^n(A_n)$, we get that $P \to M$ is an isomorphism in $D^n(A_n)$. □

Remark. As a consequence of the explicit construction, we also obtain that the resolution can be chosen so that

$$\mathrm{Hom}_{A_n}(\Gamma_i, P) \to \mathrm{Hom}_{A_n}(\Gamma_i, M)$$

is a surjection for all i; we will use this fact in Section 3.7.

We can finally apply Proposition 2.1.1 to prove the following:

Proposition 3.3.10. *The quotient functor $\mathrm{Hot}(A_n) \to D^n(A_n)$ has a fully faithful left adjoint $\mathbf{p}_n$, whose essential image is given by the n-cell modules. Moreover, the classes of n-cell modules and n-homotopy projective modules coincide.*

Proof. The first statement is a direct application of Proposition 2.1.1. For the second one, for any n-homotopy projective P take an n-cell resolution $P_n \to P$; since P is n-homotopy projective this has to be a homotopy equivalence and P is itself n-cell. □

Remark. In particular, we obtain that any n-cell A_n-module is homotopy equivalent to an n-semifree module.

Since $\mathrm{Gr}_t(-)$ and $\mathrm{Gr}_t(-)$ send n-cell modules to A-cell modules, we also get the following.

Corollary 3.3.11. *If M is an n-homotopy projective A_n-module, $\mathrm{Gr}_K(M)$ and $\mathrm{Gr}_t(M)$ are homotopy projective A-modules.*

As a partial converse, it follows from (the proof of) Lemma 3.4.1 that any graded projective A_n-module whose associated graded $\mathrm{Gr}_t(-)$ is homotopy projective is itself n-homotopy projective; a full characterization of the n-projective modules in terms of the t-adic filtration is related to the development of an appropriate obstruction theory refining [Low05], which is work in progress.

Corollary 3.3.12. *The dg subcategory $\mathcal{SF}^n(A_n) \subseteq \mathrm{Mod}\, A_n$ given by the n-semifree A_n-modules is a dg enhancement of $D^n(A_n)$.*

Proof. There is a natural functor $H^0\mathcal{SF}^n(A_n) \to D^n(A_n)$ given by the composition

$$H^0\mathcal{SF}^n(A_n) \hookrightarrow \mathrm{Hot}(A_n) \to D^n(A_n);$$

the right adjoint $\mathbf{p}_n$ gives an inverse. □

Filtered behaviors

The forgetful functor $\mathrm{Hot}(A_{n-1}) \to \mathrm{Hot}(A_n)$ carries $(n-1)$-acyclic modules to n-acyclic modules and defines a functor

$$\iota_n \colon D^{n-1}(A_{n-1}) \to D^n(A_n).$$

Corollary 3.3.13. *The functor ι_n is fully faithful.*

Proof. This follows from the fact that the forgetful functor is fully faithful and carries $(n-1)$-cell modules to n-cell modules; alternatively, one could note that any morphism from an A_{n-1}-module to an n-acyclic A_n-module M factors through the $(n-1)$-acyclic A_{n-1}-module $\operatorname{Ker} t^n_M$ and invoke [Kra09, Lemma 4.7.1]. □

Corollary 3.3.14. *For any deformation A_n, there is a system of embeddings*

$$D(A) \overset{\iota_1}{\hookrightarrow} \dots D^{n-1}(A_{n-1}) \overset{\iota_n}{\hookrightarrow} D^n(A_n).$$

This behavior is starkly different compared to the classical (dg) case, where homotopy projective modules are very much not preserved and the forgetful functor $D(A_{n-1}) \to D(A_n)$ is not fully faithful; having "all quotients at the same time" is a fundamental characteristic of this filtered setting.

3.3.2 Injective resolutions

One can dualize the argument to also obtain the existence of n-homotopy injective resolutions.

Proposition 3.3.15. *The quotient functor* $\mathrm{Hot}(A_n) \to D^n(A_n)$ *admits a fully faithful right adjoint* $\mathbf{i}_n$*, whose essential image is given by the n-homotopy injective modules.*

The construction is somewhat more involved than the projective case and is contained Section 3.3.15.

3.4 Filtered Morita equivalences

Let B_n be a cdg deformation of a dg algebra B, and let X_n be a cdg A_n-B_n bimodule. Assume that X_n is projective as a graded B_ε-module and that the A-B bimodule $X = X_n \otimes_{R_n} k$ is cofibrant as a B-module. Note that this condition is implied by cofibrancy as a bimodule.

Lemma 3.4.1. *Let X_n be an A_n-B_n bimodule that is projective as a graded B_n-module and such that $X = X_n \otimes_{R_n} k$ is cofibrant as a dg B-module. Then the functor*

$$\operatorname{Hom}_{B_n}(X_n, -)\colon \operatorname{Mod} B_n \to \operatorname{Mod} A_n$$

preserves n-acyclic modules.

Proof. Let M be a cdg B_n-module. Recall that by Proposition 3.1.2 M is n-acyclic if and only if $t^i \operatorname{Ker} t_M^{i+1}$ is acyclic for $i = 0, \dots, n$. We first show that

$$t^i \operatorname{Hom}_{B_n}(X_n, M) \cong \operatorname{Hom}_{B_n}(X_n, t^i M). \tag{3.8}$$

There is a natural map

$$t^i \operatorname{Hom}_{B_n}(X_n, M) \to \operatorname{Hom}_{B_n}(X_n, t^i M)$$

induced by the surjection $M \xrightarrow{t^i} t^i M$. This is immediately seen to be injective and, since X_n is projective as a graded B_n-module, it is also surjective. Applying now the exact functor $\operatorname{Hom}_{B_n}(X_n, -)$ to the short exact sequence

$$0 \to t^{i+1} M \to t^i M \to \operatorname{Gr}_t^i(M) \to 0$$

and using the isomorphism (3.8), we find that there is an isomorphism

$$\operatorname{Hom}_B(X, \operatorname{Gr}_t^i(M)) \cong \operatorname{Gr}_t^i(\operatorname{Hom}_{B_n}(X_n, M)).$$

If M is n-acyclic, then $\operatorname{Gr}_t^i(M)$ is acyclic; so since X is cofibrant as a B-module, the complex $\operatorname{Hom}_B(X, \operatorname{Gr}_t^i(M))$ has to be acyclic. □

Remark. (The proof of) this Lemma has some non-obvious consequences, most notably the fact that than any A_n-module that is projective as a graded A_n-module whose reduction is homotopy projective is n-homotopy projective. In particular, the $k[\varepsilon]$-module

$$\dots \to k[\varepsilon] \xrightarrow{\varepsilon} k[\varepsilon] \xrightarrow{\varepsilon} k[\varepsilon] \to \dots$$

is 1-homotopy projective.

As a consequence of this Lemma, we can derive the functor $\operatorname{Hom}_{B_n}(X_n, -)$ without the need to take an injective resolution in the second variable – morally, we have resolved X_n. In particular, any morphism of cdg algebras $A_n \to B_n$ gives to B_n the structure of a cdg A_n-B_n bimodule satisfying the hypotheses above. We have then a functor

$$\operatorname{Hom}_{B_n}(X_n, -)\colon D^n(B_n) \to D^n(A_n).$$

Proposition 3.4.2. *The functor $- \otimes_{A_n} X_n$ admits a left derived functor $- \otimes_{A_n}^{\mathrm{L}} X_n$, giving a derived adjoint pair*

$$D^n(A_n) \underset{-\otimes^{\mathrm{L}}_{A_n} X_n}{\overset{\operatorname{Hom}_{B_n}(X_n,-)}{\rightleftarrows}} D^n(B_n).$$

Proof. We define the left derived functor $-\otimes^{\mathrm{L}}_{A_n} X_n$ as the composition

$$D^n(A_n) \xrightarrow{\mathbf{p}_n} \mathrm{Hot}(A_n) \xrightarrow{-\otimes_{A_n} X_n} \mathrm{Hot}(B_n) \longrightarrow D^n(B_n).$$

To show that this is a left adjoint to $\mathrm{Hom}_{B_n}(X_n, -)$, one uses the fact that the functor $\mathrm{Hom}_{B_n}(X_n, -)$ preserves acyclic objects and consequently its left adjoint $-\otimes_{A_n} X_n$ preserves n-homotopy projective modules. □

The bimodule X_n induces A_i-B_i bimodules X_i for $i = 0, \dots, n-1$; we set $X = X_0$. The main result of this section is the following:

Proposition 3.4.3. *The adjoint pair*

$$D^n(A_n) \underset{-\otimes^{\mathrm{L}}_{A_n} X_n}{\overset{\mathrm{Hom}_{B_n}(X_n,-)}{\rightleftarrows}} D^n(B_n)$$

is an equivalence if and only if the A-B bimodule X is a Morita bimodule.

This is the union of Propositions 3.4.6 and 3.4.7, the proofs of which will take up the rest of the section. Denote with F_i the restriction of scalars $\mathrm{Mod}\, A_i \to \mathrm{Mod}\, A_n$ along the projection $A_n \to A_i$.

Lemma 3.4.4. *Let M be an A_n-module such that $t^{i+1}M = 0$. Then M has a natural structure of A_i-module and there is an isomorphism of A_n-modules*

$$M \otimes_{A_n} X_n \cong F_i(M \otimes_{A_i} X_i).$$

Proof. The first statement follows from the fact that the functor F_i identifies the category $\mathrm{Mod}\, A_i$ as the full subcategory of $\mathrm{Mod}\, A_n$ given by the modules M for which $t^{i+1}M = 0$. To prove the second, observe that since the functor F_i is fully faithful and the functor $A_i \otimes_{A_n} -$ is left adjoint to F_i there is a natural isomorphism of A_i-modules

$$M \cong M \otimes_{A_n} A_i; \tag{3.9}$$

we also know that $t^{i+1}M = 0$ implies $t^{i+1}(M \otimes_{A_n} X_n) = 0$; therefore $M \otimes_{A_n} X_n$ has the structure of a B_i-module and we can apply to it the isomorphism (3.9) to obtain a natural isomorphism of B_i-modules

$$M \otimes_{A_n} X_n \cong M \otimes_{A_n} X_n \otimes_{B_n} B_i.$$

Since the diagram

$$\begin{array}{ccc} \mathrm{Mod}\, A_n & \xrightarrow{-\otimes_{A_n} A_i} & \mathrm{Mod}\, A_i \\ {\scriptstyle -\otimes_{A_n} X_n}\downarrow & & \downarrow{\scriptstyle -\otimes_{A_i} X_i} \\ \mathrm{Mod}\, B_n & \xrightarrow{-\otimes_{B_n} B_i} & \mathrm{Mod}\, B_i \end{array}$$

commutes up to natural isomorphism, we have further isomorphisms

$$M \otimes_{A_n} X_n \cong B_i \otimes_{B_n} M \otimes_{A_n} X_n \cong M \otimes_{A_n} A_i \otimes_{A_i} X_i \cong M \otimes_{A_i} X_i$$

of B_i-modules and we are done. □

Lemma 3.4.5. *For any A_n-module M there is a natural isomorphism*

$$\mathrm{Gr}_t(M \otimes_{A_n} X_n) \cong \mathrm{Gr}_t(M) \otimes_A X.$$

Proof. We prove that there are isomorphisms

$$\mathrm{Gr}_t^i(M \otimes_{A_n} X_n) = \frac{t^i(M \otimes_{A_n} X_n)}{t^{i+1}(M \otimes_{A_n} X_n)} \cong \frac{t^i M}{t^{i+1} M} \otimes_A X = \mathrm{Gr}_t^i(M) \otimes_A X$$

for each i. By definition of the action of t^i on $M \otimes_{A_n} X_n$, we have

$$t^i(M \otimes_{A_n} X_n) \cong t^i M \otimes_{A_n} X_n.$$

so by Lemma 3.4.4 there is an isomorphism

$$t^i(M \otimes_{A_n} X_n) \cong t^i M \otimes_{A_{n-i}} X_{n-i}.$$

Since the diagram

$$\begin{tikzcd}
\mathrm{Mod}\, A_{n-i} \arrow[r, "-\otimes_{A_{n-i}} A"] \arrow[d, "-\otimes_{A_{n-i}} X_{n-i}"'] & \mathrm{Mod}\, A \arrow[d, "-\otimes_A X"] \\
\mathrm{Mod}\, B_{n-i} \arrow[r, "-\otimes_{B_{n-i}} B"] & \mathrm{Mod}\, B
\end{tikzcd}$$

commutes up to natural isomorphism, we get isomorphisms

$$\begin{aligned}
\frac{t^i(M \otimes_{A_n} X_n)}{t^{i+1}(M \otimes_{A_n} X_n)} &\cong B \otimes_{B_{n-i}} t^i(M \otimes_{A_n} X_n) \cong t^i M \otimes_{A_{n-i}} X_{n-i} \otimes_{B_{n-i}} B \\
&\cong t^i M \otimes_{A_{n-i}} A \otimes_A X \cong \frac{t^i M}{t^{i+1} M} \otimes_A X.
\end{aligned}$$

□

Proposition 3.4.6. *If X is a Morita bimodule, then*

$$D^n(A_n) \underset{-\otimes^{\mathrm{L}}_{A_n} X_n}{\overset{\mathrm{Hom}_{B_n}(X_n,-)}{\rightleftarrows}} D^n(B_n).$$

is an equivalence.

Proof. Recall that by definition, $M \otimes^{\mathrm{L}}_{A_n} X_n = \mathbf{p}_n M \otimes_{A_n} X_n$. We want to show that the unit and counit of the derived adjunction are isomorphisms; the unit is a morphism in $D^n(A_n)$

$$M \to \mathrm{Hom}_{B_n}(X_n, \mathbf{p}_n M \otimes_{A_n} X_n)$$

and the counit a morphism in $D^n(B_n)$

$$\mathbf{p}_n \,\mathrm{Hom}_{B_n}(X_n, N) \otimes_{A_n} X_n \to N;$$

The unit is represented by the roof

$$M \leftarrow \mathbf{p}_n M \to \mathrm{Hom}_{B_n}(X_n, \mathbf{p}_n M \otimes_{A_n} X_n)$$

where the left map is the canonical map $\mathbf{p}_n M \to M$ and the right is induced by the unit of the underived adjunction between $\mathrm{Hom}_{B_n}(X_n, -)$ and $- \otimes_{A_n} X_n$; the counit is given by the composition

$$\mathbf{p}_n \mathrm{Hom}_{B_n}(X_n, N) \otimes_{A_n} X_n \to \mathrm{Hom}_{B_n}(X_n, N) \otimes_{A_n} X_n \to N,$$

where the first morphism is induced by $\mathbf{p}_n \mathrm{Hom}_{B_n}(X_n, N) \to \mathrm{Hom}_{B_n}(X_n, N)$ and the second is the counit of the underived adjunction. To see that a morphism $X \to Y$ is an isomorphism in the n-derived category is is sufficient to check that $\mathrm{Gr}_t(X) \to \mathrm{Gr}_t(Y)$ is a an isomorphism in $D(A)$; the morphism induced on the associated graded by the unit is the roof

$$\begin{aligned}
&\mathrm{Gr}_t(M) \leftarrow \mathrm{Gr}_t(\mathbf{p}_n M) \to \mathrm{Gr}_t(\mathrm{Hom}_{B_n}(X_n, \mathbf{p}_n M \otimes_{A_n} X_n)) \cong \\
&\cong \mathrm{Hom}_B(X, \mathrm{Gr}_t(\mathbf{p}_n M \otimes_{A_n} X_n)) \cong \mathrm{Hom}_B(X, \mathrm{Gr}_t(\mathbf{p}_n M \otimes_A X)
\end{aligned} \tag{3.10}$$

where the first isomorphism is due to (the proof of) Lemma 3.4.1 while the second is Lemma 3.4.5. Now, since by Lemma 3.3.6 $\mathrm{Gr}_t(\mathbf{p}_n M) \to \mathrm{Gr}_t(M)$ gives a homotopy projective resolution of the A-module $\mathrm{Gr}_t(M)$, this is nothing but the unit of the adjunction induced by X between $D(A)$ and $D(B)$ applied to $\mathrm{Gr}_t(M)$: since that adjunction is an equivalence, (3.10) is an isomorphism. The argument for the case of the counit is analogous. $\square$

Proposition 3.4.7. *If*

$$D^n(A_n) \underset{-\otimes^{\mathbf{L}}_{A_n} X_n}{\overset{\mathrm{Hom}_{B_n}(X_n,-)}{\rightleftarrows}} D^n(B_n).$$

is an equivalence, then X is a Morita bimodule.

Proof. The diagrams

$$\begin{tikzcd}
D(A) \arrow[r, "\mathbf{p}"] \arrow[d, "F_0"'] & \mathrm{Hot}(A) \arrow[d, "F_0"] \\
D^n(A_n) \arrow[r, "\mathbf{p}_n"'] & \mathrm{Hot}(B_n)
\end{tikzcd} \qquad
\begin{tikzcd}
\mathrm{Mod}\, A \arrow[r, "-\otimes_A X"] \arrow[d, "F_0"'] & \mathrm{Mod}\, B \arrow[d, "F_0"'] \\
\mathrm{Mod}\, A_n \arrow[r, "-\otimes_{A_n} X_n"] & \mathrm{Mod}\, B_n
\end{tikzcd}$$

commute, the first due to the fact that any homotopy projective A-module is n-homotopy projective as an A_n-module while the second is Lemma 3.4.4. Therefore the unit and counit of the adjunction between $D(A)$ and $D(B)$ induced by X coincide with the unit and counit of the adjunction induced between $D^n(A_n)$ and $D^n(B_n)$ by X_n, applied to the modules in the image of F_0. Since F_0 is fully faithful and the latter are isomorphisms, so is the former and we are done. $\square$

3.5 Semiorthogonal decompositions

In this section we construct a semiorthogonal decomposition of the n-derived category (Theorem 3.5.7). Ideally we would like each of the factors to be generated by one of the objects Γ_i defined in Section 3.2.2, but it turns out that these have nontrivial morphisms in both directions and we need to slightly tweak the generators. We define

$$G_n = \mathrm{coCone}(\Gamma_n \to \frac{\Gamma_n}{t^n \Gamma_n}).$$

Lemma 3.5.1. *For any A_n-module M, there is a quasi-isomorphism*

$$\mathrm{Hom}_{A_n}(G_n, M) \cong t^n M. \tag{3.11}$$

In particular, G_n is n-homotopy projective.

Proof. One sees that, as graded modules,

$$\mathrm{Hom}_{A_n}(\Gamma_n, M) \cong \mathrm{Ker}\, t^n_M[-1] \oplus M \text{ and } \mathrm{Hom}_{A_n}\left(\frac{\Gamma_n}{t^n\Gamma_n}, M\right) \cong \mathrm{Ker}\, t^n_M[-1] \oplus \mathrm{Ker}\, t^n_M$$

with predifferentials induced by the usual twisting matrix (3.2). The functor $\mathrm{Hom}_{A_n}(-, M)$ sends cocones to cones, so there is an isomorphism

$$\mathrm{Hom}_{A_n}(G_n, M) \cong \mathrm{Cone}(\mathrm{Hom}_{A_n}\left(\frac{\Gamma_n}{t^n\Gamma_n}, M\right) \overset{J}{\hookrightarrow} \mathrm{Hom}_{A_n}(\Gamma_n, M))$$

where J is given by the identity in the first component and the inclusion $\mathrm{Ker}\, t^n_M \hookrightarrow M$ in the second. Since J is injective, there is a natural quasi-isomorphism $\mathrm{Cone}\, J \cong \mathrm{Coker}\, J$; looking at the explicit form of $\mathrm{Coker}\, J$, one sees that it is isomorphic to $M/\mathrm{Ker}\, t^n_M \cong t^n M$ not only as a graded module but also as complex, since no component of the matrix (3.2) twists the component of the predifferential going from the second factor to itself. □

Recall from Proposition 3.3.13 that we had the embedding

$$\iota_n \colon D^{n-1}(A_{n-1}) \hookrightarrow D^n(A_n)$$

induced by the restriction functor: denote by $\iota_n D^{n-1}(A_{n-1}) \subseteq D^n(A_n)$ its essential image. Its left adjoint $M \to \frac{M}{t^n M}$ defines a functor

$$\mathrm{Coker}\, t^n \colon D^n(A_n) \to D^{n-1}(A_{n-1})$$

which, as a consequence of the fact that the restriction functor carries $(n-1)$-acyclics to n-acyclics, is still a left adjoint to ι_n.

Lemma 3.5.2. *The subcategory $\iota_n D^{n-1}(A_{n-1}) \subseteq D^n(A_n)$ is given by the modules M for which $t^n M$ is acyclic.*

Proof. For any module in the image of the restriction functor one has $t^n M = 0$, and since isomorphisms in $D^n(A_n)$ induce quasi-isomorphisms on $\mathrm{Im}\, t^n$ it's clear that all the modules M in $\iota_n D^{n-1}(A_{n-1})$ have the desired property. If instead M is such that $t^n M$ is acyclic, the map $M \to \frac{M}{t^n M}$ is an isomorphism in the n-derived category between M and an element in the image of the restriction functor - M and $\frac{M}{t^n M}$ have the same graded pieces with respect to the t-filtration, with the exception of M having the extra piece $t^n M$. □

Definition 3.5.3. Define the triangulated subcategory $\mathcal{T}_n \subseteq D^n(A_n)$ as given by the modules $M \in D^n(A_n)$ for which $\frac{M}{t^n M}$ is $(n-1)$-acyclic.

It follows directly from the definition that $\mathcal{T}_n$ is a localizing subcategory. By the same argument as Lemma 3.5.2, the subcategory $\mathcal{T}_0$ coincides with $\iota_n \iota_{n-1} \dots \iota_1 D(A)$.

Proposition 3.5.4. *There is a semiorthogonal decomposition*

$$D^n(A_n) = \langle \iota_n D^{n-1}(A_{n-1}), \mathcal{T}_n \rangle$$

and $\mathcal{T}_n$ coincides with the minimal localizing subcategory of $D^n(A_n)$ containing the module G_n.

Proof. First we show that there are no nonzero morphisms from $\mathcal{T}_n$ to $\iota_n D^{n-1}(A_{n-1})$; let $M \in \mathcal{T}_n$ and $N \in \iota_n D^{n-1}(A_{n-1})$ and assume that $N \cong \iota_n N'$ for some $N' \in D^{n-1}(A_{n-1})$. We have

$$\mathrm{Hom}_{D^n(A_n)}(M, N) \cong \mathrm{Hom}_{D^n(A_n)}(M, \iota_n N') \cong \mathrm{Hom}_{D^{n-1}(A_{n-1})}\left(\frac{M}{t^n M}, N'\right)$$

which is 0 since M lies in $\mathcal{T}_n$. To verify that this is a semiorthogonal decomposition, we construct for any M a triangle

$$P_G \to M \to N \to P_G[1]$$

where P_G is in $\mathcal{T}_n$ and N is in $D^{n-1}(A_{n-1})$. For this, apply Proposition 3.3.8 to M with $X = G_n$ to obtain a G_n-cell module P_G with a closed morphism $P_G \to M$ inducing a quasi-isomorphism $t^n P_G \cong t^n M$; denoting with N its cone we know that $t^n N$ is acyclic, so N lies in $D^{n-1}(A_{n-1})$. Finally,

$$\frac{G_n}{t^n G_n} \cong \mathrm{coCone}\left(\frac{\Gamma_n}{t^n \Gamma_n} \xrightarrow{\mathrm{id}} \frac{\Gamma_n}{t^n \Gamma_n}\right)$$

is contractible so G_n lies in $\mathcal{T}_n$. Since $\mathcal{T}_n$ is localizing, any G_n-cell module lies in $\mathcal{T}_n$; this also proves the second claim. ☐

Proposition 3.5.5. *The functor* $\mathrm{Im}\, t^n \colon D^n(A_n) \to D(A)$ *induces an equivalence*

$$\begin{tikzcd} \mathcal{T}_n \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & D^n(A_n) \arrow[r, "\mathrm{Im}\, t^n"] & D(A). \end{tikzcd}$$

In particular, the functor $\mathrm{Im}\, t^n$ *identifies* $D(A)$ *with the quotient* $\frac{D^n(A_n)}{\iota_n D^{n-1}(A_{n-1})}$.

Proof. The functor is essentially surjective since it preserves coproducts and it sends $G_n \in \mathcal{T}_n$ to the generator $A \in D(A)$, so we only have to verify full faithfulness; since G_n is a compact generator of $\mathcal{T}_n$ and it is sent to a compact object, it is enough (by the same classical argument as e.g. [LO10, Proposition 1.15]) to show that

$$\mathrm{Hom}_{A_n}(G_n, G_n) \to \mathrm{Hom}_A(A, A)$$

is a quasi-isomorphism; this will follow once we prove the commutativity of the square

$$\begin{tikzcd} \mathrm{Hom}_{A_n}(G_n, G_n) \arrow[r] \arrow[d, "\varphi"'] & \mathrm{Hom}_A(A, A) \arrow[d, "\wr"] \\ t^n G_n \arrow[r, "\sim"] & A \end{tikzcd}$$

where the right and lower arrows are the obvious isomorphisms, and φ is the quasi-isomorphism of Lemma 3.5.1. To see the explicit form of the morphism φ, recall that G_n has a graded submodule (not preserved by the predifferential) $A_n \subseteq G_n$ corresponding to the graded submodule $A_n \subseteq \Gamma_n$; denote by $1_n \in G_n$ the element corresponding to the unit of A_n. The morphism φ is then given explicitly by

$$\begin{aligned} \mathrm{Hom}_{A_n}(G_n, M) &\to t^n M \\ [f\colon G_n \to M] &\to t^n f(1_n); \end{aligned}$$

in this form, it is clear that the square commutes and we are done. $\square$

Remark. In fact it is very easy to see that $\iota_n D^{n-1}(A_{n-1}) \subseteq D^n(A_n)$ is an admissible subcategory: the right adjoint to the embedding is given by the functor $\operatorname{Ker} t^n$ and the left adjoint by $\operatorname{Coker} t^n$; using the description of the quotient $\frac{D^n(A_n)}{\iota_n D^{n-1}(A_{n-1})}$ given in Proposition 3.5.5, we see that there is a recollement

$$\begin{tikzcd} D^{n-1}(A_{n-1}) \arrow[r, hook, "\iota_n"] & D^n(A_n) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "\operatorname{Im} t^n"] & D(A). \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

Definition 3.5.6. Define the subcategories $\mathcal{T}_i \subseteq D^n(A_n)$ for $i = 0, \dots, n$ as given by the modules M for which all the graded components $\mathrm{Gr}_t^j(M)$ for $j \neq i$ are acyclic.

By the same argument as Lemma 3.5.2, the subcategory $\mathcal{T}_0$ coincides with $\iota_1 D(A)$.

Theorem 3.5.7. *The n-derived category $D^n(A_n)$ admits a semiorthogonal decomposition*

$$D^n(A_n) = \langle \mathcal{T}_0, \mathcal{T}_1, \dots \mathcal{T}_n \rangle$$

and the functors

$$\mathrm{Gr}_t^i(-)\colon D^n(A_n) \to D(A)$$

induce equivalences $\mathcal{T}_i \xrightarrow{\sim} D(A)$.

Proof. This follows by iterating the arguments of Propositions 3.5.4 and 3.5.5; the only further thing to check is that the intersection of $\iota_n D^{n-1}(A_{n-1})$ with the subcategory of the modules M for which $\mathrm{Gr}_t^{n-1}(M)$ is $(n-2)$-acyclic is given as claimed by the modules M for which $\mathrm{Gr}_t^i(M)$ is acyclic for $i \neq n-1$, but this follows from reasoning as in Lemma 3.5.2. $\square$

In particular if A_n is a first order deformation of A, there is a short exact sequence of categories[3]

$$0 \to D(A) \hookrightarrow D^1(A_1) \overset{M \to tM}{\to} D(A) \to 0$$

categorifying the usual square zero extension

$$0 \to A \hookrightarrow A_1 \to A \to 0.$$

corresponding to the deformation A_1.

3.6 Relation with the semiderived category

In [Pos11; Pos18], Positselski defines several notions of acyclicity for cdg modules; we recall those, and explain the relation to ours (Corollary 3.6.8).

3.6.1 The semiderived category

Definition 3.6.1. ([Pos11]) A cdg A_n-module is said to be *absolutely acyclic* if it lies in the minimal thick subcategory of $\mathrm{Hot}(A_n)$ containing all totalizations of short exact sequences; it is *coacyclic* if it lies in the minimal triangulated subcategory of $\mathrm{Hot}(A_n)$ which contains all totalizations of short exact sequences and is closed under arbitrary coproducts, and *contraacyclic* if it lies in the minimal triangulated subcategory of $\mathrm{Hot}(A_n)$ which contains totalizations of short exact sequences and is closed under arbitrary products.

One key property of contraacyclic modules is that they are right orthogonal to the graded projective A_n-modules; dually, coacyclic modules are left orthogonal to the graded injective A_n-modules ([Pos11, Theorem 3.5.1]).

Definition 3.6.2. ([Pos18]) An R_n-free A_n-module M is said to be *semiacyclic* if M/tM is an acyclic A-module. An arbitrary A_n-module is *semiacyclic as a comodule* if it lies in the minimal thick subcategory of $\mathrm{Hot}(A_n)$ containing all coacyclic modules and all R_n-free semiacyclic modules, and *semiacyclic as a contramodule* if it lies in the minimal thick subcategory of $\mathrm{Hot}(A_n)$ containing all contraacyclic modules and all R_n-free semiacyclic modules. Define the category $D^{\mathrm{si}}(A_n^{R_n-\mathrm{fr}})$ as the quotient of the homotopy category of R_n-free modules by the semiacyclic A_n-modules, the category $D^{\mathrm{si}}(A_n^{\mathrm{co}})$ as the quotient of $\mathrm{Hot}(A_n)$ by modules which are semiacyclic as comodules and $D^{\mathrm{si}}(A_n^{\mathrm{contra}})$ as the quotient of $\mathrm{Hot}(A_n)$ by the modules which are semiacyclic as contramodules.

There are natural functors

$$D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \to D^{\mathrm{si}}(A_n^{\mathrm{contra}}) \text{ and } D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \to D^{\mathrm{si}}(A_n^{\mathrm{co}})$$

induced by the inclusion of the homotopy category of R_n-free A_n-modules into $\mathrm{Hot}(A_n)$ which are shown in [Pos18, Theorem 4.2.1] to be equivalences; one refers to either version interchangeably as the *semiderived category* of A_n, which is then denoted with $D^{\mathrm{si}}(A_n)$. In particular, it follows from [Pos18, Theorem 4.2.1] that an R_n-free A_n-module is semiacyclic as a co/contramodule if and only if it is semiacyclic as an R_n-free module.

[3] By this we mean that the first arrow is fully faithful and the second is a quotient whose kernel is given by the essential image of the first one.

Remark. In the definitions of [Pos18], the different versions of the semiderived categories are defined starting from categories of comodules and contramodules; however, since the rings R_n are artinian every module is both a comodule and a contramodule, and the only difference lies in the class of acyclics.

There is a fundamental difference between $D^n(A_n)$ and the categories that we just presented: since in the semiderived category - and in general, in all the derived categories considered in [Pos11] and [Pos18] - the absolutely acyclic modules are quotiented out, short exact sequences give rise to triangles in the quotient. This crucially is not the case in $D^n(A_n)$, where it is easy to give examples of absolutely acyclic modules which are not n-acyclic; consider again the simple case $A = k$, $A_1 = R_1$. Then the R_1-module

$$M = 0 \to k \to R_1 \to k \to 0$$

is absolutely acyclic but not 1-acyclic. There are nonetheless relations between the two notions, as we now show.

Lemma 3.6.3. *An R_n-free module M is semiacyclic if and only if it is n-acyclic.*

Proof. Obviously any n-acyclic R_n-free module is semiacyclic. For the converse, assume M to be semiacyclic and consider the short exact sequence

$$0 \to \frac{\operatorname{Ker} t^i_M}{\operatorname{Ker} t^i_M \cap tM} \to \frac{M}{tM} \xrightarrow{t^i} \frac{t^i M}{t^{i+1} M} \to 0;$$

since M is R_n-free, $\operatorname{Ker} t^i_M = t^{n-i+1} M$ and $\operatorname{Ker} t^i_M \cap tM = \operatorname{Ker} t^i_M$ so $\operatorname{Gr}^i_t(M) \cong M/tM$ for all i, and M is n-acyclic.

□

As a consequence, the inclusion from the homotopy category of R_n-free A_n-modules into $\operatorname{Hot}(A_n)$ carries semiacyclic modules to n-acyclic modules and defines a functor

$$\varphi \colon D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \to D^n(A_n).$$

Free resolutions

In [Pos18], the author constructs for any A_n-module M a triangle in $\operatorname{Hot}(A_n)$

$$M^{\mathrm{fr}} \to M \to C \to M^{\mathrm{fr}}[1] \tag{3.12}$$

where C is contraacyclic and M^{fr} is R_n-free. Concretely, he builds an exact sequence

$$\ldots \to F_1 \to F_0 \to M \to 0$$

where each F_i is R_n-free, and defines

$$M^{\mathrm{fr}} = \operatorname{Tot}^{\Pi}(\ldots \to F_1 \to F_0 \to 0)$$

so that

$$C = \operatorname{Tot}^{\Pi}(\ldots \to F_1 \to F_0 \to M \to 0)$$

is contraacyclic by [Pos18, Lemma 4.2.2]. Note that, since for the ring R_n the classes of free and cofree modules coincide, a product of free modules is still a free module. For the same reason, any R_n-module admits an injective map into an R_n-free module and the argument above dualizes to obtain a triangle

$$B \to M \to M^{\mathrm{cofr}} \to B[1]$$

with

$$M^{\mathrm{cofr}} = \mathrm{Tot}^{\oplus}(0 \to F'_0 \to F'_1 \to \dots)$$

where each F'_i is R_n-free such that

$$B = \mathrm{Tot}^{\oplus}(0 \to M \to F'_0 \to F'_1 \to \dots)$$

is coacyclic.

3.6.2 Derived functors of the reductions

Consider the functor

$$\mathrm{Coker}\, t\colon \mathrm{Mod}\, A_n \to \mathrm{Mod}\, A,$$

and denote it with Q for simplicity.

The categories $\mathrm{Mod}\, A_n$ and $\mathrm{Mod}\, A$ are dg categories, but their underlying categories $Z^0(\mathrm{Mod}\, A_n)$ and $Z^0(\mathrm{Mod}\, A)$ have a natural abelian structure. Seen as a functor between abelian categories, the functor Q is right exact and as such we are interested in its left derived functors. Although we have not shown the category $Z^0(\mathrm{Mod}\, A_n)$ to have enough projectives, R_n-free modules are adapted to Q and there are enough R_n-free modules, so the left derived functors exist. The next proposition gives a proof of this fact by explicitly describing the higher derived functors.

Proposition 3.6.4. *The functor Q admits left derived functors L^iQ defined as*

$$L^iQ(M) = \begin{cases} Q(M) \text{ for } i = 0, \\[2ex] \dfrac{\mathrm{Ker}\, t_M}{t^n M} \text{ for } i \text{ odd}, \\[2ex] \dfrac{\mathrm{Ker}\, t^n_M}{tM} \text{ for } i > 0 \text{ and even.} \end{cases}$$

Proof. We show that, by setting $L^0Q = Q$ and L^iQ as above, the collection L^*Q gives a universal homological δ-functor. Let

$$0 \to M \to N \to L \to 0$$

be a short exact sequence of A_n-modules. Consider, for any A_n-module K, the complex of A_n-modules

$$K^t = \dots \to K \xrightarrow{t} K \xrightarrow{t^n} K \xrightarrow{t} K \to 0.$$

By definition, one has $H^0K^t = Q(K)$, $H^{-i}K^t = L^iQ(K)$ for $i > 0$. Since the morphisms $M \to N$ and $N \to L$ are morphisms of A_n-modules, they commute with the action of t and induce a short exact sequence of complexes

$$0 \to M^t \to N^t \to L^t \to 0.$$

The long exact sequence in homology reads

$$\ldots \to \frac{\operatorname{Ker} t_L^n}{tL} \to \frac{\operatorname{Ker} t_M}{t^n M} \to \frac{\operatorname{Ker} t_N}{t^n N} \to \frac{\operatorname{Ker} t_L}{t^n L} \to \frac{M}{tM} \to \frac{N}{tN} \to \frac{L}{tL} \to 0.$$

Since everything is functorial, we have proven that our candidate derived functors give rise to a δ-functor. To show that it is universal, we use the fact that it is coeffaceable: indeed, if an A_n-module M is R_n-free then $\frac{\operatorname{Ker} t_M}{t^n M} = \frac{\operatorname{Ker} t_M^n}{tM} = 0$ and, as shown in the proof of [Pos18, Theorem 4.2.1], any cdg A_n-module admits a surjection from an R_n-free module. □

Since $L^iQ(F) = 0$ for all R_n-free modules F and $i > 0$, to compute $L^i(M)$ it is enough to take an R_n-free resolution

$$\ldots \to F_1 \to F_0 \to M \to 0,$$

apply the functor Q term-wise and read the functor $L^iQ(M)$ as the horizontal cohomology of the complex of A-modules

$$\ldots \to Q(F_1) \to Q(F_0) \to 0.$$

Deriving the right adjoint

Denote by K the functor $\operatorname{Ker} t\colon \operatorname{Mod} A_n \to \operatorname{Mod} A$. The functor K is left exact, and the arguments of the previous sections dualize to show that K admits right derived functors.

Proposition 3.6.5. *The functor K admits right derived functors R^iK, and one has*

$$R^iK(M) = \begin{cases} K(M) \text{ for } i = 0, \\ R^iK(M) = \dfrac{\operatorname{Ker} t_M^n}{tM} \text{ for } i \text{ odd}, \\ \dfrac{\operatorname{Ker} t_M}{t^n M} \text{ for } i > 0 \text{ and even.} \end{cases}$$

Peculiarly, it turns out that the right derived functors of K coincide up to a shift in periodicity with the left derived functors of Q.

We can now prove the main result of this section.

Proposition 3.6.6. *Any n-acyclic A_n-module is semiacyclic both as a comodule and as a contramodule.*

Proof. Assume that M is an n-acyclic A_n-module, and consider the triangle (3.12). Since C is contraacyclic it is in particular semiacyclic as a contramodule, and in order to prove that M is semiacyclic it will be enough to prove that

$$M^{\mathrm{fr}} = \mathrm{Tot}^{\Pi}(\ldots \to F_1 \to F_0 \to 0)$$

is semiacyclic. This, since M^{fr} is R_n-free, is equivalent to $Q(M^{\mathrm{fr}})$ being acyclic. Clearly

$$Q(M^{\mathrm{fr}}) \cong \mathrm{Tot}^{\Pi}(\ldots \to Q(F_1) \to Q(F_0) \to 0)$$

and by [EM62, Proposition 6.2] we can compute the cohomology of the total complex by taking first horizontal and then vertical cohomology. The horizontal cohomology computes the derived functors $L^iQ(M)$, so the first page reads

$$\ldots \qquad L^2Q(M) \qquad L^1Q(M) \qquad L^0Q(M).$$

Since M is n-acyclic we know that $L^0Q(M) \cong Q(M)$ is acyclic; similarly, it is straightforward to see that since M is n-acyclic $L^iQ(M)$ is also acyclic for $i > 0$ so we are done. The argument for semiacyclicity as a comodule is dual. □

Remark. If A_n has no curvature one can give a different proof of this fact, at least for semiacyclicity as a contramodule. First of all observe that, since M is n-acyclic, it is enough to prove that $Q(C)$ is acyclic to conclude that $Q(M^{\mathrm{fr}})$ is acyclic. Then, consider the dg A_n-module

$$L_n = \mathrm{Tot}^{\oplus}(0 \to A_n \xrightarrow{t} A_n \xrightarrow{t^n} A_n \xrightarrow{t} A_n \to \ldots).$$

The module L_n is A_n-free, so by [Pos11, Theorem 3.5.1] since C is contraacyclic we know that $\mathrm{Hom}_{A_n}(L_n, C)$ is acyclic. Explicitly, we have

$$\mathrm{Hom}_{A_n}(L_n, M) \cong \mathrm{Tot}^{\Pi}(\ldots \to C \xrightarrow{t} C \xrightarrow{t^n} C \xrightarrow{t} C \to 0)$$

and we can again compute its cohomology by taking first horizontal and then vertical cohomology. The first page is

$$\frac{\mathrm{Ker}\, t_C}{t^nC} \qquad \frac{\mathrm{Ker}\, t_C^n}{tC} \qquad \frac{\mathrm{Ker}\, t_C}{t^nC} \qquad \frac{C}{tC}$$

with the natural vertical differentials. Since M is n-acyclic and M^{fr} is R_n-free one has that $\frac{\mathrm{Ker}\, t_C}{t^nC}$ and $\frac{\mathrm{Ker}\, t_C^n}{tC}$ are acyclic; the second page then reads

$$0 \qquad 0 \qquad 0 \qquad H^{\bullet}\frac{C}{tC}$$

so $\frac{C}{tC}$ has to be acyclic. This proof does not generalize well to the curved case due to the difficulty of finding explicit noncontractible R_n-free cdg A_n-modules.

We have then proven the following:

Corollary 3.6.7. *The semiderived category $D^{\mathrm{si}}(A_n)$ is a quotient of the n-derived category $D^n(A_n)$.*

Explicitly, the semiderived category $D^{\mathrm{si}}(A_n^{\mathrm{contra}})$ is the quotient of $D^n(A_n)$ by (the closure under filtered quasi-isomorphisms of) the contraacyclic modules; similarly the semiderived category $D^{\mathrm{si}}(A_n^{\mathrm{co}})$ is the quotient of $D^n(A_n)$ by the closure of the coacyclic modules.

Corollary 3.6.8. *The quotient functor $D^n(A_n) \to D^{\mathrm{si}}(A_n^{\mathrm{contra}})$ admits a left adjoint* **fr**, *defined by assigning to a module M its free resolution M^{fr}. Dually, the quotient functor $D^n(A_n) \to D^{\mathrm{si}}(A_n^{\mathrm{co}})$ admits a right adjoint* **cf**, *defined by assigning to a module M its cofree resolution M^{cofr}.*

Proof. We prove the first statement, the second being dual. The claim will follow once we prove that $\mathrm{Hom}_{D^n(A_n)}(F, S) = 0$ as soon as F is R_n-free and S is semiacyclic as a contramodule, since at that point we will be able to use the triangle (3.12) to apply Proposition 2.1.1. Any morphism $F \to S$ in $D^n(A_n)$ can be represented as a roof $F \to M \xleftarrow{\sim} S$ where both arrows are morphisms in $\mathrm{Hot}(A_n)$ and $S \to M$ has n-acyclic cone; since S is semiacyclic as a contramodule, by Proposition 3.6.6 so is M. We will then be done if we prove that any closed morphism between an R_n-free A_n-module and a module which is semiacyclic as a contramodule factors through an n-acyclic module. The proof of this fact is essentially contained in [Pos18]; indeed by the discussion in the proof of [Pos18, Theorem 4.2.1] the class $\mathcal{F}$ of modules M for which any closed morphism $F \to M$ from an R_n-free module factors through an R_n-free n-acyclic module is closed under cones; it is also shown there that any morphism $F \to C$ where C is contraacyclic factors through an R_n-free contraacyclic module, and since any R_n-free contraacyclic module is n-acyclic, all contraacyclic modules lie in $\mathcal{F}$. Finally, all R_n-free n-acyclic modules lie in $\mathcal{F}$ – just take the trivial factorization $F \to N \xrightarrow{\mathrm{id}} N$ – so the class $\mathcal{F}$ contains all the modules that are semiacyclic as contramodules and we are done. □

Recall the functor $\varphi\colon D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \to D^n(A_n)$ defined after Lemma 3.6.3.

Corollary 3.6.9. *The functor φ coincides both with the composition*

$$D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \xrightarrow{\sim} D^{\mathrm{si}}(A_n^{\mathrm{contra}}) \xrightarrow{\mathbf{fr}} D^n(A_n)$$

as well as with the composition

$$D^{\mathrm{si}}(A_n^{R_n\text{-fr}}) \xrightarrow{\sim} D^{\mathrm{si}}(A_n^{\mathrm{co}}) \xrightarrow{\mathbf{cf}} D^n(A_n).$$

In particular, it is fully faithful and admits both a left and a right adjoint.

Proof. This follows from the fact that if M is R_n-free, then one can take $M^{\mathrm{fr}} = M^{\mathrm{cofr}} = M$. □

In other words, $D^{\mathrm{si}}(A_n)$ identifies with the R_n-free part of $D^n(A_n)$. It is straightforward to see that the functors L^iQ carry n-acyclic modules to acyclic modules, and thus define functors $D^n(A_n) \to D(A)$. It turns out that we can fully characterize the semiderived category in terms of the functors L^iQ. Let us assume for simplicity that $n = 1$.

Proposition 3.6.10. *The subcategory $D^{\mathrm{si}}(A_1) \subseteq D^1(A_1)$ coincides with the kernel of the functor $L^1Q\colon D^1(A_1) \to D(A)$.*

Proof. We have already shown that the subcategory $D^{\mathrm{si}}(A_1) \subseteq D^n(A_1)$ coincides with the class of all modules which, up to filtered quasi-isomorphism, are R_1-free. If $M \in \operatorname{Mod} A_1$ is R_1-free, it is clear that $L^1Q(M) = 0$. Vice versa, assume that $L^1Q(M)$ is acyclic. Then, via the usual R_1-free resolution, we have an R_1-free module M^{fr} with a morphism $M^{\mathrm{fr}} \to M$ whose cone is isomorphic to

$$C = \operatorname{Tot}^{\Pi}(\ldots \to F_1 \to F_0 \to M).$$

We have that

$$Q(C) \cong \operatorname{Tot}^{\Pi}(\ldots \to Q(F_1) \to Q(F_0) \to Q(M)).$$

Since $L^1Q(M)$ is acyclic and $L^1Q(M) = L^iQ(M)$ for all $i > 0$, by the same spectral sequence argument as in Proposition 3.6.6, $Q(C)$ is acyclic. Then, since $L^1Q(M^{\mathrm{fr}}) = 0$, we get that $L^1Q(C) = \frac{\operatorname{Ker} t_C}{tC}$ is acyclic. Finally, by the short exact sequence

$$0 \to \frac{\operatorname{Ker} t_C}{tC} \to \frac{C}{tC} \to tC \to 0$$

we get that C is 1-acyclic and $M^{\mathrm{fr}} \to M$ is an isomorphism in $D^1(A_1)$. □

The case $n > 1$ is analogous, with the semiderived category coinciding with the intersection of the kernels of the first derived functors of the reductions $\operatorname{Coker} t^i \colon \operatorname{Mod} A_n \to \operatorname{Mod} A_{i-1}$. We omit the proof.

Remark. One might wonder whether the two subcategories $D^{\mathrm{si}}(A_n)$ and $D^{n-1}(A_{n-1})$ generate $D^n(A_n)$ in any way. This is indeed the case for uncurved deformations, but not in general. As is explained in [Pos18, Example 5.3.6] the semiderived category of the deformation $k_u[u, u^{-1}]$ from Example 3.2.2 is the zero category; on the other hand, we know from Theorem 3.5.7 that $D^1(k_u[u, u^{-1}])$ cannot coincide with its subcategory $\iota_1 D(k[u, u^{-1}])$ since their quotient is nontrivial. In fact, one can prove that if the semiderived category is “large enough”, i.e. there exists a module $M \in D^{\mathrm{si}}(A_n^{R_n\text{-fr}})$ such that $Q(M)$ is a compact generator of $D(A)$, then the deformation A_n is appropriately equivalent to an uncurved deformation (see [LVdB15] for a similar argument). This perspective will be further developed in future work.

3.7 A model structure

In this section we show the existence of a cofibrantly generated model structure on $Z^0(\operatorname{Mod} A_n)$ presenting the n-derived category, which generalizes the classical projective model structure; just like the construction of the resolutions, this will pass through a relative version of the relevant classical construction. The proofs in this section follow closely those presented in [CH02], which hold in the case of complexes of objects in an abelian category; the main points in which our proofs deviate from [CH02] are the proofs of Lemmas 3.7.7 and 3.7.8, together with Proposition 3.7.10.

Let $\mathcal{X}$ be a set of finitely presented cdg A_n-modules; since each $X \in \mathcal{X}$ is finitely presented, $\operatorname{Hom}_{A_n}(X, -)$ commutes with directed colimits.

Definition 3.7.1. A closed morphism $f \colon M \to N$ is an $\mathcal{X}$*-equivalence* if

$$f_* \colon \operatorname{Hom}_{A_n}(X, M) \to \operatorname{Hom}_{A_n}(X, N)$$

is a quasi-isomorphism for all $X \in \mathcal{X}$.

If M is any cdg A_n-module, denote by C_M the module $\mathrm{Cone}(\mathrm{id}_M)$, so that there is a natural closed morphism $M \to C_M$. Our first goal is to show the following:

Proposition 3.7.2. *The category $Z^0(\mathrm{Mod}\, A_n)$ admits a cofibrantly generated model structure where the weak equivalences are the $\mathcal{X}$-equivalences, the fibrations are the morphisms $f\colon M \to N$ for which*

$$f_*\colon \mathrm{Hom}_{A_n}(X, M) \to \mathrm{Hom}_{A_n}(X, N)$$

is surjective for all $X \in \mathcal{X}$ and the cofibrations are the maps with the left lifting property with respect to the acyclic fibrations. The generating cofibrations and generating acyclic cofibrations are given by the sets

$$I = \{X[l] \to C_X[l]\}_{X\in\mathcal{X}, l\in\mathbb{Z}} \quad \text{and} \quad J = \{0 \to C_X[l]\}_{X\in\mathcal{X}, l\in\mathbb{Z}}.$$

The argument is similar to the proof of [CH02, Theorem 5.7] and follows from an application of the so-called recognition lemma; to state that, we will need the following definition.

Definition 3.7.3 ([Hov99, Definition 2.1.7])**.** Let S be a set of morphisms in a category $\mathcal{C}$. Define the class S-inj as given by the morphisms with the right lifting property with respect to each map in S; the class S-cof as given by the morphisms with the left lifting property with respect to the maps in S-inj, and the class S-cell as given by the maps obtained as transfinite compositions of pushouts of coproducts of maps in S.

Note that in general, $S\text{-cell} \subseteq S\text{-cof}$. We can now state the recognition lemma.

Proposition 3.7.4 ([Hov99, Theorem 2.1.19])**.** *Let $\mathcal{C}$ be a complete and cocomplete category, let W be a class of morphisms that is closed under retracts and satisfies the two-out-of-three property, and let I and J be sets of morphisms whose domains are small with respect to (respectively) I-cell and J-cell such that:*

- $I\text{-inj} = J\text{-inj} \cap W$;
- $J\text{-cell} \subseteq I\text{-cof} \cap W$.

Then $\mathcal{C}$ admits a cofibrantly generated model structure with I as generating cofibrations, J as generating acyclic cofibrations, and W as weak equivalences.

In particular, it follows from [Hov99, Proposition 2.1.18] that the cofibrations are the retracts of maps in I-cell and the acyclic cofibrations are retracts of maps in J-cell. To apply Proposition 3.7.4 in our setting, we will need the following lemma.

Lemma 3.7.5. *Let I and J be as in Proposition 3.7.2. A closed morphism $A \xrightarrow{f} B$ has the right lifting property with respect to the maps in J if and only if the map*

$$f_*\colon \mathrm{Hom}_{A_n}(X, A) \to \mathrm{Hom}_{A_n}(X, B)$$

is surjective for all $X \in \mathcal{X}$; it has the right lifting property with respect to the maps in I if and only if f_* is a surjective quasi-isomorphism.

Proof. Let $X \in \mathcal{X}$. We first prove the statement about J; a diagram of the form

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C_X & \longrightarrow & B \end{array}$$

corresponds to the choice of a (not necessarily closed) degree 0 map $X \xrightarrow{x} B$; a lift then corresponds precisely to a map $X \to A$ which is sent to x via f_*; hence, f_* is surjective in degree 0. To see that this holds in arbitrary degree, just consider the various shifts of the maps in I. In the rest of this section we will omit this "shifting" step, and will usually not mention the degree of the various maps. The converse is shown by the same argument.

Let us now show that having the right lifting property relative to I implies that f_* is a surjective quasi-isomorphism; assume that $A \xrightarrow{f} B$ has the right lifting property relative to I. First, we prove that f_* is surjective on closed maps: any closed morphism $X \xrightarrow{x} B$ defines a diagram

$$\begin{array}{ccc} X & \xrightarrow{0} & A \\ \downarrow & & \downarrow f \\ C_X & \xrightarrow{(x,0)} & B. \end{array}$$

A lift $C_X \to A$ then defines a closed map $X \to A$ mapping to x via f_*. To see that f_* is surjective on arbitrary morphisms, let $X \xrightarrow{y} B$ be a (not necessarily closed) map; then dy is closed, and by the previous step there exists a closed map $X \xrightarrow{x} A$ which lifts dy. Hence the diagram

$$\begin{array}{ccc} X & \xrightarrow{x} & A \\ \downarrow & & \downarrow f \\ C_X & \xrightarrow{(y,dy)} & B \end{array}$$

commutes, and a lift of this diagram yields a lift of $X \to A$ of y. Since f_* is surjective on cycles it's also surjective in cohomology, so all that's left to show is that it's injective in cohomology. Given a closed map $X \xrightarrow{x} A$ such that $fx = dy$ for some $B \xrightarrow{y} A$, the diagram

$$\begin{array}{ccc} X & \xrightarrow{x} & A \\ \downarrow & & \downarrow f \\ C_X & \xrightarrow{(y,dy)} & B \end{array}$$

commutes. Now a lift provides a map $X \xrightarrow{z} A$ such that $dz = x$, i.e. f_* must be injective in cohomology. To show the converse, we need to show that if f_* is a surjective quasi-isomorphism then any diagram of the form

$$\begin{array}{ccc} X & \xrightarrow{x} & A \\ \downarrow & & \downarrow f \\ C_X & \xrightarrow{(y,dy)} & B \end{array}$$

admits a lift; commutativity here corresponds to the fact that $fx = dy$. A lift corresponds to a map $X \xrightarrow{z} A$ such that $dz = x$ and $fz = y$. Since f_* is injective in cohomology, there exists a map $X \xrightarrow{z_1} A$ such that $dz_1 = x$. Now $y - fz_1$ may not be zero, but since $d(y - fz_1) = dy - fdz_1 = fx - fx = 0$, it is a closed map; hence, there exists a closed map $X \xrightarrow{z_2} A$ such that $fz_2 = y - fz_1$. Therefore, $z = z_1 + z_2$ gives the desired lift. $\square$

Proof of Proposition 3.7.2. Take as W the class of $\mathcal{X}$-equivalences and I, J as already described. The category $Z^0(\operatorname{Mod} A_n)$ is both complete and cocomplete, and it is clear that W is closed under retracts and satisfies that two-out-of-three property. Since the domains of the morphisms in I and J are finitely presented, they are small relative to any class of morphisms. It follows immediately from Lemma 3.7.5 that $I\text{-inj} \cap W = J\text{-inj}$, so the only thing left to prove is that $J\text{-cell} \subseteq I\text{-cof} \cap W$. Since C_X is contractible, a pushout of coproducts of maps in J will always be of the form $X \to X \oplus C$ for some contractible module C; hence, the same can be said for a transfinite compositions of map of this kind. This is always a homotopy equivalence, so in particular an $\mathcal{X}$-equivalence. Again by Lemma 3.7.5 we know that $I\text{-inj} \subseteq J\text{-inj}$ so $J\text{-cof} \subseteq I\text{-cof}$; since $J\text{-cell} \subseteq J\text{-cof}$, we get that $J\text{-cell} \subseteq I\text{-cof}$ and by Proposition 3.7.4 we are done. $\square$

Applying Proposition 3.7.2 to the set $\mathcal{X} = \{\Gamma_0, \dots, \Gamma_n\}$ we obtain

Proposition 3.7.6. *The category $Z^0(\operatorname{Mod} A_n)$ admits a cofibrantly generated model structure where the weak equivalences are the n-quasi-isomorphisms. The fibrations are the morphisms $f\colon M \to N$ such that the induced map $\operatorname{Ker} t^i_M \to \operatorname{Ker} t^i_N$ is surjective for all $i \in 1, \dots, n+1$.*

Proof. To verify the description of the fibrations, the only thing that must be checked is that a morphism $f\colon M \to N$ is a fibration for the model structure given by Proposition 3.7.2 if and only if the induced morphism $\operatorname{Ker} t^i_M \to \operatorname{Ker} t^i_N$ is surjective for all i. This follows immediately from the isomorphisms of graded modules

$$\operatorname{Hom}_{A_n}(\Gamma_0, M) \cong \operatorname{Ker} t_M \text{ and } \operatorname{Hom}_{A_n}(\Gamma_i, M) \cong \operatorname{Ker} t^{i+1}_M \oplus \operatorname{Ker} t^i_M[-1] \text{ for } i > 0.$$

$\square$

We are left with characterizing the cofibrations.

Lemma 3.7.7. *Any cofibrant A_n-module is n-homotopy projective; that is, any closed map from a cofibrant module to a n-acyclic module admits a nullhomotopy.*

Proof. Let Q be a cofibrant module. If N is an n-acyclic module then the natural morphism $C_N[-1] \to N$ is an acyclic fibration, and since Q is cofibrant any morphism $Q \to N$ admits an extension $Q \to C_N[-1]$ i.e. a nullhomotopy. $\square$

Lemma 3.7.8. *Given a fibration $M \to N$, any arbitrary (not necessarily closed) morphism $Q \to M$ with a cofibrant domain admits*[4] *a lift $Q \to N$.*

Proof. This follows from the fact that arbitrary morphisms $Q \to M$ correspond to closed morphisms $Q \to C_M$; since C_M is contractible, the map $C_M \to C_N$ induced by $M \to N$ is an acyclic fibration and since Q is cofibrant the required lift exists. $\square$

[4]This is the appropriate version of the notion of graded projectivity.

Proposition 3.7.9. *In the model structure of Proposition 3.7.6, the cofibrations are the graded split monomorphisms with cofibrant cokernel.*

Proof. Let $A \xrightarrow{i} B$ be a cofibration; to construct a splitting, observe that the object C_A is contractible and, therefore, the terminal map $C_A \to 0$ is an acyclic fibration. Now a lift to the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] & C_A \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

gives a (not necessarily closed) splitting for i. To see that a cofibration has cofibrant cokernel, we use that those, by [Hov99, Corollary 1.1.11], are closed under pushouts. Since cokernels are pushouts, any cofibrations has cofibrant cokernel. To show the converse, we need to prove that any map $A \xrightarrow{i} B$ which is a graded split monomorphism with cofibrant cokernel has the left lifting property with respect to any acyclic fibration $X \xrightarrow{p} Y$. Choosing a splitting for i we can write B as $\mathrm{Cone}(\tau)$ for some closed map $C \xrightarrow{\tau} A$, where C is the cofibrant cokernel of i; this way, i corresponds to the inclusion $A \to \mathrm{Cone}(\tau)$. Consider the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d, "p"] \\ \mathrm{Cone}(\tau) \arrow[r, "g"] & Y. \end{tikzcd}$$

The map g corresponds to a pair (pf, α), with α a morphism $C \to Y$ with the property that $d\alpha = pf\tau$; a lift h of g will then be given by a pair (f, β) where $C \xrightarrow{\beta} X$ is a map with the property that $p\beta = \alpha$ and $d\beta = f\tau$. Using Lemma 3.7.8, we can lift α to a (not necessarily closed) map $C \xrightarrow{\gamma} X$ with the property that $p\gamma = \alpha$. Denote with $K \xrightarrow{j} X$ the kernel of p; setting $\delta = d\gamma - f\tau$, one sees that $p\delta = 0$ and thus δ lifts to a closed map $C \xrightarrow{F} K$ such that $jF = \delta$. Since K is n-acyclic, by Lemma 3.7.7 there exists a map $C \xrightarrow{D} K$ such that $dD = F$. The map $\beta = \gamma - jD$ now defines the desired lift. $\square$

Proposition 3.7.10. *A cdg A_n-module is cofibrant according to the model structure described in Proposition 3.7.6 if and only if it is a retract of an n-semifree module.*

Proof. The modules Γ_i and their shifts are cofibrant because they are the cokernels of the generating cofibrations, thus so is an arbitrary coproduct of them. If P is an n-semifree module, by Proposition 3.7.9 the inclusions $F_iP \hookrightarrow F_{i+1}P$ are cofibrations so their transfinite composition $0 \hookrightarrow P$ is also a cofibration, hence P is cofibrant. Vice versa if Q is cofibrant, the resolution $P_Q \to Q$ given by Corollary 3.3.9 is readily seen to be an acyclic fibration, and we know that P_Q is n-semifree. Since Q is cofibrant, the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & P_Q \arrow[d] \\ Q \arrow[r, "\mathrm{id}_Q"] & Q \end{tikzcd}$$

admits a lift, i.e. Q is a retract of P_Q.

$\square$

It is well-known that the category $Z^0(\operatorname{Mod} A_n)$ admits a cofibrantly generated model structure with weak equivalences given by the quasi-isomorphisms and fibrations given by the surjective morphisms. For that model structure, both the adjunctions

$$\operatorname{Mod} A \underset{F}{\overset{\operatorname{Ker} t}{\leftrightarrows}} \operatorname{Mod} A_n \qquad \text{and} \qquad \operatorname{Mod} A \underset{\operatorname{Coker} t}{\overset{F}{\rightleftarrows}} \operatorname{Mod} A_n$$

are Quillen adjunctions.

Proposition 3.7.11. *The model structure on $Z^0(\operatorname{Mod} A_n)$ is obtained by right transfer from the one on $Z^0(\operatorname{Mod} A)$ along the adjunction*

$$\operatorname{Mod} A \underset{\operatorname{Coker} t}{\overset{F}{\rightleftarrows}} \operatorname{Mod} A_n.$$

Proof. We have to prove that a closed morphism $f\colon M \to N$ in $Z^0(\operatorname{Mod} A)$ is a fibration or a weak equivalence precisely when its image via the forgetful functor is. It is immediate to see that the forgetful functor preserves and reflects weak equivalences, so the only question is about the fibrations; this follows from the fact that if M is in the image of F, then $\operatorname{Ker} t_M^i = M$ for all $i > 0$. □

3.8 The formal case

In this last section we consider the case of a formal deformation A_t over the formal power series ring $k[[t]]$; it is well known [AJL97; DG02; PSY14; Pos16; LVdB15] that for formal deformations the correct category to consider is not the classical derived category. Indeed the theory developed in the artinian case does not generalize verbatim to the formal case because in general, the formal analogue of Lemma 3.1.2 does not hold: indeed for any torsionfree A_t-module M – take for example any $k[[t]]$-free module – one has $\operatorname{Ker} t_M^i = 0$ for all i, so no information can be inferred on $\operatorname{Gr}_t(M)$ just by knowing $\operatorname{Gr}_K(M)$. Similarly, for any t-divisible A_t-module – for example, any $k[[t]]$-cofree[5] module – one has that $\operatorname{Gr}_t(M) = 0$ regardless of the acyclicity of $\operatorname{Gr}_K(M)$. Therefore to get a meaningful theory we have to place ourselves in a setting where some (homological) form of Nakayama's lemma holds. In this paper we restrict to studying the simpler torsion case, although we do expect that a parallel construction of a filtered derived category of $k[[t]]$-contramodule A_t-modules (a version of the complete derived category) is also possible, together with an appropriate version of the co-contra correspondence.

Definition 3.8.1. A $k[[t]]$-module M is *torsion* if for every $m \in M$ there exists an n such that $t^n m = 0$. An A_t-module is torsion if it is torsion as an $k[[t]]$-module. The dg category $\operatorname{Mod} {A_t}^{\mathrm{tor}}$ is the full dg subcategory of $\operatorname{Mod} A_t$ having as objects the torsion A_t-modules; we will denote by $\operatorname{Hot}(A_t)^{\mathrm{tor}}$ the corresponding triangulated homotopy category; since a coproduct of torsion modules is still a torsion module, $\operatorname{Hot}(A_t)^{\mathrm{tor}}$ is a triangulated category with arbitrary coproducts.

If $k[t]$ is the coalgebra whose linear dual is the algebra $k[[t]]$, $k[t]$-comodules coincide with torsion $k[[t]]$-modules. For torsion modules a version of Nakayama's lemma holds: if $\operatorname{Ker} t_M$ is the zero module, then so is M.

[5]The relevant notion of cofreeness is that of [Pos18]; the prototypical example of a cofree module is the $k[[t]]$-module $k((t))/k[[t]]$.

Definition 3.8.2. A torsion A_t-module is said to be *t-acyclic* if $\mathrm{Gr}_K(M)$ is acyclic; it is said to be t-homotopy projective if $\mathrm{Hom}_{A_t}(M,N)$ is acyclic for any t-acyclic torsion module N.

The following should be understood as a homological version of Nakayama's lemma for comodules/torsion modules.

Proposition 3.8.3. *If M is a t-acyclic torsion A_t-module, then $\mathrm{Gr}_t(M)$ is acyclic.*

Proof. We first prove that $\frac{M}{tM}$ is acyclic; let $x = m + tM \in \frac{M}{tM}$ be a closed element, i.e. such that $dm = tk$ for some n. Since M is torsion, there exists an integer N such that $m, k \in \mathrm{Ker}\, t_M^N$. Since M is t-acyclic, $\mathrm{Ker}\, t_M^N$ is $(N-1)$-acyclic so, in particular, $\frac{\mathrm{Ker}\, t_M^N}{t \,\mathrm{Ker}\, t_M^N}$ is acyclic; being $m + t\,\mathrm{Ker}\, t_M^N$ a closed element in $\frac{\mathrm{Ker}\, t_M^N}{t\,\mathrm{Ker}\, t_M^N}$, there exists an $l \in \mathrm{Ker}\, t_M^N$ such that $m - d(l) \in t\,\mathrm{Ker}\, t_M^N \subseteq tM$; hence $\frac{M}{tM}$ is acyclic. To conclude that all the graded pieces are acyclic we can reason by induction using the short exact sequence

$$0 \to \frac{t^{i-1}\,\mathrm{Ker}\, t_M^i}{t^i\,\mathrm{Ker}\, t_M^{i+1}} \to \frac{t^{i-1}M}{t^iM} \xrightarrow{t} \frac{t^iM}{t^{i+1}M} \to 0.$$

□

In particular, this tells us that we could have defined a module M to be t-acyclic if both $\mathrm{Gr}_t(M)$ and $\mathrm{Gr}_K(M)$ are acyclic. On the other hand, since any $k[[t]]$-cofree module is both torsion and divisible, there are plenty of examples of torsion modules for which the converse does not hold.

Definition 3.8.4. The *t-derived category* of torsion modules $D^t(A_t)^{\mathrm{tor}}$ is defined as the quotient of the homotopy category $\mathrm{Hot}(A_n)^{\mathrm{tor}}$ by the t-acyclic modules.

Since $\mathrm{Gr}_K(-)$ commutes with coproducts, the torsion t-derived category is a triangulated category with arbitrary coproducts.

Proposition 3.8.5. *The modules $\Gamma_0, \Gamma_1, \ldots \in \mathrm{Mod}\, A_t{}^{\mathrm{tor}}$ are t-homotopy projective compact generators of the category $D^t(A_t)^{\mathrm{tor}}$.*

Proof. The deformation A_t induces by truncation infinitesimal deformations $A_n = \frac{A_t}{t^{n+1}A_t}$ of any order $n \geq 0$; like in the infinitesimal case, there is a fully faithful restriction dg functor

$$A_n\,\mathrm{Mod} \to A_t\,\mathrm{Mod}^{\mathrm{tor}}$$

which has a right adjoint constructed by assigning to a torsion cdg A_t-module M the A_n-module $\mathrm{Ker}\, t_M^{n+1}$. Essentially by definition, a torsion A_t-module M is t-acyclic if and only if $\mathrm{Ker}\, t_M^{n+1}$ is n-acyclic for all $n > 0$. Hence if M is t-acyclic, one gets

$$\mathrm{Hom}_{A_t}(\Gamma_i, M) \cong \mathrm{Hom}_{A_i}(\Gamma_i, \mathrm{Ker}\, t_M^{i+1})$$

which is acyclic since Γ_i is homotopy projective as an A_i-module; hence, each Γ_i is t-homotopy projective. Since the functor $\mathrm{Ker}\, t^j$ for any $j \geq 0$ commutes with coproducts, they are also compact (see also the proof of Proposition 3.2.4). To see that they generate

$D^t(A_t)^{\mathrm{tor}}$, observe that if $\mathrm{Hom}_{A_t}(\Gamma_i, M)$ is acyclic for all i, then for any $n \geq 0$ and $i \leq n$ one has that

$$\mathrm{Hom}_{A_t}(\Gamma_i, M) \cong \mathrm{Hom}_{A_n}(\Gamma_i, \mathrm{Ker}\, t_M^{n+1})$$

is acyclic, and by Theorem 3.2.5 this implies that $\mathrm{Ker}\, t_M^{n+1}$ is n-acyclic. Therefore since $\mathrm{Ker}\, t_M^{n+1}$ is n-acyclic for all n, the A_t-module M must t-acyclic and we are done. □

In other words, the category $D^t(A_t)^{\mathrm{tor}}$ is the colimit in the category of presentable ∞-categories (which in this case is the closure under colimits of the naive colimit) of the system of embeddings

$$D(A) \hookrightarrow D_1(A_1) \hookrightarrow \ldots \hookrightarrow D^n(A_n) \hookrightarrow \ldots$$

Remark. Like in the artinian case, Proposition 3.8.5 implies that the quotient $\mathrm{Hot}(A_n)^{\mathrm{tor}} \to D^t(A_n)^{\mathrm{tor}}$ has a fully faithful left adjoint.

The definition of the semiderived category $D^{\mathrm{si}}(A_t^{\mathrm{co}})$ with comodule coefficients in the formal setting is essentially the same as the one given in Section 3.6: it is defined as the quotient of the homotopy category of $k[[t]]$-cofree torsion A_t-modules by the subcategory given by the modules with acyclic reduction ([Pos18, Section 2.3]). This can also be shown to be equivalent to the quotient of the homotopy category of torsion A_t-modules by the subcategory given by the modules that are semiacyclic as comodules ([Pos18, Section 4.3]).

Proposition 3.8.6. *There is a left admissible embedding $D^{\mathrm{si}}(A_t^{\mathrm{co}}) \hookrightarrow D^t(A_t)^{\mathrm{tor}}$.*

Proof. This has essentially the same proof as in the artinian case, the only modification being the fact that the right derived functors of the left exact functor

$$\mathrm{Ker}\, t \colon \mathrm{Mod}\, {A_t}^{\mathrm{tor}} \to \mathrm{Mod}\, A$$

are now $R^1K(M) = \frac{M}{tM}$ and $R^iK = 0$ for $i > 1$. To see this, one uses the existence for every short exact sequence

$$0 \to M \to N \to L \to 0$$

of the short exact sequence

$$0 \to \mathrm{Ker}\, t_M \to \mathrm{Ker}\, t_N \to \mathrm{Ker}\, t_L \to \frac{M}{tM} \to \frac{N}{tN} \to \frac{L}{tL} \to 0$$

and the fact that any torsion A_t-module admits an inclusion into a $k[[t]]$-cofree, and thus divisible, A_t-module. □

In the case where A_t is a dg algebra then, as discussed in [Pos18, Section 0.16], a complex of $k[[t]]$-cofree $k[[t]]$-modules has acyclic reduction if and only if it is itself acyclic. Note that one implication – that is, the fact that acyclicity implies acyclicity of the reduction – relies crucially on the fact that the algebra $k[[t]]$ is regular; see Example 3.1.1 for a case where that implication fails in a singular setting. Hence, the semiderived category with comodule (torsion) coefficients coincides with the classical derived category of torsion modules; note that the restriction to $k[[t]]$-cofree modules does not create issues, since any (torsion) module admits a $k[[t]]$-cofree resolution. It then follows that the (torsion) derived category of A_t can be identified with a left admissible subcategory of $D^t(A_t)^{\mathrm{tor}}$.

3.9 Construction of the injective resolutions

In this section we prove the existence of n-homotopy injective resolutions of A_n-modules (Proposition 3.9.5). By definition, an n-homotopy injective resolution of an A_n-module M is given by an n-homotopy injective module I equipped with a n-quasi-isomorphism $M \to I$. Just like in the classical case this is a bit more delicate than the projective case, and will require some auxiliary constructions.

Left modules

Everything that we have proven up until this point for right A_n-modules also holds, with opportune modifications (mostly of signs), for left A_n-modules; denote by $A_n \operatorname{Mod}$ the dg category of left cdg A_n-modules. We have $n+1$ left A_n-modules $D_0 \dots D_n$ - defined in the same way as the right modules Γ_i except for the fact that $\frac{c}{t}$ acts on the right in the twisting matrix - which generate the n-derived category of left A_n-modules. In this setting, n-cell modules are those built out of the modules D_i and like in the case of right modules, any left module M admits an n-cell resolution $P \to M$.

Linear duality

Denote by $\operatorname{Mod}\mathbb{Z}$ the dg category of complexes of abelian groups, and as usual with $\operatorname{Hom}_{\mathbb{Z}}(X, Y)$ the complex of $\mathbb{Z}$-linear morphisms; define the linear duality functor $(-)^\vee$ as $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. One sees that if M is a left A_n-module, $M^\vee$ has a natural structure of a right A_n-module and vice versa. Since $\mathbb{Q}/\mathbb{Z}$ is injective as an abelian group, the functor $(-)^\vee$ is exact; using in addition the fact that $M^\vee = 0$ implies $M = 0$, we get that the linear duality functor reflects exactness.

Applying the functor $(-)^\vee$ to the exact sequence

$$0 \to \operatorname{Ker} f \xrightarrow{f} M \to N \to \operatorname{Coker} f \to 0$$

one obtains, for any morphism f, natural isomorphisms

$$\operatorname{Ker} f^\vee \cong (\operatorname{Coker} f)^\vee \text{ and } (\operatorname{Ker} f)^\vee \cong \operatorname{Coker} f^\vee. \tag{3.13}$$

If M is a - either left or right - A_n-module, there is a canonical evaluation morphism

$$\begin{aligned} \operatorname{ev}_M \colon M &\to M^{\vee\vee} = (M^\vee)^\vee \\ m &\to [\eta \to \eta(m)] \end{aligned}$$

which is easily seen to be an injective closed morphism of A_n-modules. Moreover, as a consequence of the isomorphisms (3.13), there is a natural isomorphism $(\operatorname{Coker} d_M)^{\vee\vee} \cong \operatorname{Coker} d_{M^{\vee\vee}}$ under which the map $\operatorname{ev}_{\operatorname{Coker} d_M}$ corresponds to the map induced by ev_M between the cokernels of the differentials. Since $\operatorname{ev}_{\operatorname{Coker} d_M}$ is injective, we get that ev_M induces an injective map between the cokernels of the differential.

Definition 3.9.1. Define the right A_n-modules $\Gamma_i^* = D_i^\vee$. A right A_n-module is said to be *n-cocell* if it lies in the minimal triangulated subcategory of $\operatorname{Hot}(A_n)$ containing $\Gamma_0^*, \dots, \Gamma_n^*$ which is closed under products.

It is straightforward to see that if an A_n-module P is n-cell, then $P^\vee$ is n-cocell.

Auxiliary functors

Define the functors $F_i\colon A_n\,\mathrm{Mod} \to \mathrm{Mod}\,\mathbb{Z}$ and $Q_i\colon A_n\,\mathrm{Mod} \to \mathrm{Mod}\,\mathbb{Z}$ as

$$F_i(M) = \mathrm{Hom}_{A_n}(D_i, M),\ Q_i(M) = D_i \otimes_{A_n} M;$$

note that the tensor product of a right cdg module with a left cdg module is by definition a complex. Explicitly, $F_i(M)$ is given by the qdg module $\mathrm{Ker}\, t_M^{i+1} \oplus \mathrm{Ker}\, t_M^i[-1]$ twisted by the matrix (3.2) while $Q_i(M)$ is the qdg module $\frac{M}{t^{i+1}M} \oplus \frac{M}{t^iM}[1]$ twisted by the matrix

$$\begin{bmatrix} 0 & \pi \circ \frac{c}{t} \\ t & 0 \end{bmatrix}$$

where $t\colon \frac{M}{t^{i+1}M} \to \frac{M}{t^iM}$ is induced by the action of t, and $\pi\colon \frac{M}{t^iM} \to \frac{M}{t^{i+1}M}$ is the natural projection.

Lemma 3.9.2. *An A_n-module M is n-acyclic if and only if the A_n-module $M^\vee$ is n-acyclic.*

Proof. Using again the isomorphism (3.13) one sees that the linear duality functor exchanges the graded pieces of the t-adic filtration and the K-filtration, giving isomorphisms

$$\left(\frac{t^iM}{t^{i+1}M}\right)^\vee = \left(\mathrm{Ker}\, \frac{M}{t^{i+1}M} \to \frac{M}{t^iM}\right)^\vee \cong \mathrm{Coker}(\mathrm{Ker}\, t^i_{M^\vee} \to \mathrm{Ker}\, t^{i+1}_{M^\vee}) = \frac{\mathrm{Ker}\, t^{i+1}_{M^\vee}}{\mathrm{Ker}\, t^i_{M^\vee}} \tag{3.14}$$

and

$$\left(\frac{\mathrm{Ker}\, t^{i+1}_M}{\mathrm{Ker}\, t^i_M}\right)^\vee = \left(\mathrm{Coker}(\mathrm{Ker}\, t^i_M \to \mathrm{Ker}\, t^{i+1}_M)\right)^\vee \cong \mathrm{Ker}\left(\frac{M^\vee}{t^{i+1}M^\vee} \to \frac{M^\vee}{t^iM^\vee}\right) = \frac{t^iM^\vee}{t^{i+1}M^\vee}. \tag{3.15}$$

If M is n-acyclic, by (3.14) and the fact that $(-)^\vee$ is exact we get that $M^\vee$ is also n-acyclic. The other implication follows from the same argument together with fact that $(-)^\vee$ reflects exactness. □

Lemma 3.9.3. *The functors F_i and Q_i have the following properties:*

1. *There are natural isomorphisms*
$$F_i(M)^\vee \cong Q_i(M^\vee) \text{ and } Q_i(M)^\vee \cong F_i(M^\vee). \tag{3.16}$$

2. *Under the isomorphism $Q_i(M)^{\vee\vee} \cong F_i(M^\vee)^\vee \cong Q_i(M^{\vee\vee})$ the map*
$$Q_i(\mathrm{ev}_M)\colon Q_i(M) \to Q_i(M^{\vee\vee})$$
corresponds to
$$\mathrm{ev}_{Q_i(M)}\colon Q_i(M) \to Q_i(M)^{\vee\vee}.$$
In particular, $Q_i(\mathrm{ev}_M)$ is injective and induces an injection between the cokernels of the differentials.

3. *The functors Q_i are right exact and preserve products;*

4. An A_n-module M is n-acyclic if and only if $Q_i(M)$ is acyclic for all i.

Proof. The proofs of points 1) and 2) consist only in writing down the explicit forms of F_i and Q_i and repeatedly applying the isomorphisms (3.13). The fact that Q_i preserves products follows again from the fact that A_i is finitely presented as an A_n-module, so the functor $A_i \otimes_{A_n} -$ preserves products. Since F_i is defined as a hom-functor it is left exact, and the fact that Q_i is right exact follows then from point 1) together with the fact that the linear duality functors preserves and reflects exactness: this proves point 3). Finally, point 4) follows again from point 2) together with the facts that the modules D_i are n-homotopy projective generators of the n-derived category and Lemma 3.9.2. □

Lemma 3.9.4. *Any n-cocell module is n-homotopy injective.*

Proof. This will follow once we prove that the modules $\Gamma_i^* = D_i^\vee$ are n-homotopy injective. For that, we use that there are isomorphisms

$$\mathrm{Hom}_{A_n}(M, D_i^\vee) \cong \mathrm{Hom}_{A_n}(D_i, M^\vee)$$

defined by assigning to a morphism $f\colon M \to D_i^\vee$ the morphism

$$\begin{aligned} D_i &\to M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \\ r &\to [m \to f(m)(r)]. \end{aligned}$$

The claim then follows from Lemma 3.9.2. □

As before, an n-cocell resolution is a closed morphism of A_n-modules $M \to I$ where I is n-cocell with n-acyclic cone; it is clear that an n-cocell resolution is in particular an n-homotopy injective resolution.

Construction of the injective resolutions

We are now ready to prove the following fact:

Proposition 3.9.5. *Any A_n-module M admits an n-cocell resolution $M \to I$.*

Proof. Consider the left A_n-module $M^\vee$; recalling that $F_i(-) = \mathrm{Hom}_{A_n}(D_i, -)$, apply to $M^\vee$ the version for left modules of Lemma 3.3.8 with $\mathcal{X} = \{D_0, \dots D_n\}$ to obtain a closed morphism

$$P \xrightarrow{p} M^\vee$$

with $P \in n$-cell such that $F_i(P) \overset{F_i(p)}{\to} F_i(M)$ and $\mathrm{Ker}\, d_{F_i(P)} \overset{F_i(p)}{\to} \mathrm{Ker}\, d_{F_i(M)}$ are surjective for all i. Dualizing, we get a map

$$M^{\vee\vee} \xrightarrow{p^\vee} P^\vee$$

with the property that $F_i(M^\vee)^\vee \overset{F_i(p)^\vee}{\to} F_i(P)^\vee$ and $(\mathrm{Ker}\, d_{F_i(M^\vee)})^\vee \overset{F_i(p)^\vee}{\to} (\mathrm{Ker}\, d_{F_i(M^\vee)})^\vee$ are injections. Using Lemma 3.9.3 1) and the fact that under the isomorphism (3.16) we have $F_i(p)^\vee = Q_i(p^\vee)$, we get that $p^\vee$ induces injections $Q_i(M^{\vee\vee}) \overset{Q_i(p^\vee)}{\to} Q_i(P^\vee)$ and $\operatorname{Coker} d_{Q_i(M^{\vee\vee})} \overset{Q_i(p^\vee)}{\to} \operatorname{Coker} d_{Q_i(P^\vee)}$. Compose now $p^\vee$ with ev_M to obtain a map

$$M \overset{\mathrm{ev}_M}{\to} M^{\vee\vee} \overset{p^\vee}{\to} P^\vee.$$

Set $P^\vee = I_0$; since ev_M induces injections on Q_i and on the cokernels of the differentials, we have that the induced maps $Q_i(M) \to Q_i(I_0)$ and $\operatorname{Coker} d_{Q_i(M)} \to \operatorname{Coker} d_{Q_i(I_0)}$ are injective. Moreover since P is n-cell, $I_0 = P^\vee$ is n-cocell. Denote by C_0 the cokernel of $M \to I_0$; since Q_i and $\operatorname{Coker} d_{Q_i(-)}$ are right exact functors, the sequences

$$0 \to Q_i(M) \to Q_i(I_0) \to Q_i(C_0) \to 0$$

and

$$0 \to \operatorname{Coker} d_{Q_i(M)} \to \operatorname{Coker} d_{Q_i(I_0)} \to \operatorname{Coker} d_{Q_i(C_0)} \to 0$$

are exact. Applying again the procedure described above to obtain a morphism $C_0 \to I_1$, we can iterate the construction to get a sequence

$$M \to I_0 \to I_1 \to I_2 \to \dots$$

such that each I_k is n-cocell and the sequences

$$0 \to Q_i(M) \to Q_i(I_0) \to Q_i(I_1) \to Q_i(I_2) \to \dots$$

and

$$0 \to \operatorname{Coker}_{Q_i(M)} \to \operatorname{Coker}_{Q_i(I_0)} \to \operatorname{Coker}_{Q_i(I_1)} \to \operatorname{Coker}_{Q_i(I_2)} \to \dots$$

are exact. Setting $I = \mathrm{Tot}^\Pi(I_\bullet)$, we have a natural map $M \to I$ whose cone is isomorphic to

$$T = \mathrm{Tot}^\Pi(0 \to M \to I_0 \to I_1 \to I_2 \to \dots);$$

Since Q_i commutes with cones and products, we get that

$$Q_i(T) \cong \mathrm{Tot}^\Pi(0 \to Q_i(M) \to Q_i(I_0) \to Q_i(I_1) \to Q_i(I_2) \to \dots)$$

which by [Sta, 09J0] is acyclic; therefore T is n-acyclic, and $M \to I$ is an isomorphism in $D^n(A_n)$. The proof that I is n-cocell is completely dual to the projective case, consisting of an application of the dual of Lemma 3.3.3 (see [Sta, 09KR]). □

This concludes the proof of Proposition 3.3.15. We also obtain:

Corollary 3.9.6. *The classes of homotopy injective and n-cocell modules coincide. In particular if M is an n-homotopy injective A_n-module, then $\mathrm{Gr}_t(M)$ and $\mathrm{Gr}_K(M)$ are homotopy injective A-modules.*

Remark. The fact that our construction for injective resolutions is somewhat more involved than the one for projective resolutions is due to the fact that, while the generators Γ_i are compact, the cogenerators Γ_i^* are *not* cocompact - cocompact objects rarely exist in module categories. This is why we had to introduce the functors Q_i; indeed while the complex $\mathrm{Hom}_{A_n}(\prod_k M_k, \Gamma_i^*) \cong Q_i(\prod_k M_k)^\vee$ seems hard to control, the fault lies only in the linear dual - once we manage to get rid of it, the functor Q_i is as well behaved as one could hope.

Just like we could have proven the existence of projective resolutions using Brown representability, it is possible to use a similar strategy for injective resolutions; however, since as we discussed the cogenerators Γ_i^* are not cocompact, some care is needed. The correct ingredient to use would seem to be the notion of 0-compactness introduced in [OPS19] which is strictly related to the fact that, while it is not possible to write down $\mathrm{Hom}_{A_n}(\prod_k M_k, \Gamma_i^*)$ in terms of $\mathrm{Hom}_{A_n}(M_k, \Gamma_i^*)$, it is true that if $\bigoplus_k \mathrm{Hom}_{A_n}(M_k, \Gamma_i)$ is acyclic the same is true for $\mathrm{Hom}_{A_n}(\prod_k M_k, \Gamma_i^*)$ and vice versa. Using the fact that the modules Γ_i^* are 0-cocompact, Theorem 6.6 of [OPS19] yields the existence of the desired resolution.

Chapter

4

Extensions of triangulated categories

In Chapter 3 we introduced the 1-derived category of a first order deformation A_1 of A, and showed that it admits a semiorthogonal decomposition that leads to considering it as a deformation of the derived category of $D(A)$. The goal of this chapter is to formalize this fact, and give a definition of (first order) deformation) of a triangulated category which encompasses the case of the n-derived category. The idea is the obvious one: we want to define a deformation of a triangulated category $\mathcal{T}$ as a category $\mathcal{T}_\varepsilon$ admitting a recollement

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "i"] & \mathcal{T}_\varepsilon \arrow[l, bend right, "Q"'] \arrow[l, bend left, "K"] \arrow[r, "I"] & \mathcal{T}. \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

This alone cannot be a meaningful definition: there are too many ways to glue together two copies of $\mathcal{T}$ – essentially, as many as there are functors $\mathcal{T} \to \mathcal{T}$. Any recollement identifies a gluing functor and any functor can be used to construct a gluing (see Section 2.3.4). The idea is that we want the gluing functor to be the cone of a certain Hochschild class, seen as a natural transformation

$$\mathrm{id}_{\mathcal{T}}[-1] \to \mathrm{id}_{\mathcal{T}}[1].$$

Any Hochschild class defines a (noncanonical) functor $\mathcal{T} \to \mathcal{T}$ by choosing a cone in the (triangulated) functor category, but in general it is impossible to recover from an object in a triangulated category a morphism admitting it as a cone; some extra data is needed. For us, that extra data comes in the form of an (appropriately defined) Yoneda extension of functors

$$0 \to I \to K \to Q \to I \to 0$$

abstracting the exact sequence

$$0 \to tM \to \operatorname{Ker} t_M \to M/tM \to tM \to 0$$

defined for any A_1-module M. Defining thus a categorical deformation of a triangulated category $\mathcal{T}$ as a recollement together with a compatible Yoneda extension, we are able to show the main result of this chapter (Theorem 4.1.9): any Hochschild class yields a deformation, any deformation defines an Hochschild class and the two construction are inverses to each other. We therefore obtain the desired bijection between HH^2 and first order deformations.

4.1 Categorical deformations

In this section we introduce the notion of categorical deformation of a triangulated category (Definition 4.1.1) and show how to any deformation corresponds an Hochschild class. From this point onward, we will implicitly assume that all triangulated categories come with an enhancement; hence, we will use the word triangulated synonym as a synonym to pretriangulated dg category. An exception to this rule will be given by derived categories of bimodules and quasi-functors: in that case, by $D(\mathcal{A}^{\mathrm{op}} \otimes \mathcal{B})$ we will mean the *homotopy category* of the relevant enhancement, and with $\mathrm{Hom}_{D(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})}(M, N)$ the k-module of morphisms in the derived category. To highlight this fact, we will at times also use the notation $\mathrm{Ext}^i_{\mathrm{Fun}(\mathcal{A},\mathcal{B})}(M, N)$ to signify $\mathrm{Hom}_{D(\mathcal{A}^{\mathrm{op}}\otimes\mathcal{B})}(M, N[i])$. In the same vein, isomorphisms between quasi-functors will always mean isomorphisms in the homotopy category.

4.1.1 Definition and basic properties

Definition 4.1.1. Let $\mathcal{T}$ be a pretriangulated dg category. A *first order categorical deformation* of $\mathcal{T}$ consists of the following data:

1. A recollement $\mathcal{T}_\varepsilon$ as in

$$\begin{tikzcd} \mathcal{T} \arrow[r, hook, "i"] & \mathcal{T}_\varepsilon \arrow[l, bend right, "Q"'] \arrow[l, bend left, "K"] \arrow[r, "I"] & \mathcal{T} \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

   with associated $\alpha\colon K \to Q$;

2. A Yoneda 2-extension from I to I compatible with the semiorthogonal decomposition, i.e. whose splicing sequence is of the form

$$0 \to E \xrightarrow{\delta_1} K \xrightarrow{\alpha} Q \xrightarrow{\delta_2} E \to 0 \tag{4.1}$$

   for some δ_1, δ_2.

In other words, the deformation is given by a recollement together with two triangles

$$E \xrightarrow{\delta_1} K \xrightarrow{\zeta_1} C \xrightarrow{\eta_1} E[1] \text{ and } C \xrightarrow{\eta_2} Q \xrightarrow{\delta_2} E \xrightarrow{\zeta_2} C[1]$$

with the property that the composition

$$K \xrightarrow{\zeta_1} C \xrightarrow{\eta_2} Q$$

coincides with α as a map in the derived category. We will routinely abuse notation and denote with $\mathcal{T}_\varepsilon$ the pair of recollement and extension.

From this data, we easily get a Hochschild class: we have a map $\iota\colon E[-1] \to E[1]$ defined as the composition of the two boundary maps $E[-1] \xrightarrow{\zeta_2[-1]} C \xrightarrow{\eta_1} E[1]$; and we can compose

ι on the right with G to obtain a natural transformation $EG[-1] \stackrel{\iota G}{\to} EG[1]$; recalling now the natural isomorphism $\mathrm{id}_{\mathcal{T}} \stackrel{g}{\to} EG$, we obtain the class

$$\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon) \in \mathrm{HH}^2(\mathcal{T}) = \mathrm{Ext}^2_{\mathrm{Fun}(\mathcal{T},\mathcal{T})}(\mathrm{id}_{\mathcal{T}}, \mathrm{id}_{\mathcal{T}})$$

as the unique morphism $\mathrm{id}_{\mathcal{T}}[-1] \to \mathrm{id}_{\mathcal{T}}[1]$ in $D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$ fitting in the commutative diagram

$$\begin{tikzcd}
EG[-1] \arrow[r, "\iota G"] & EG[1] \\
\mathrm{id}_{\mathcal{T}}[-1] \arrow[u, "g"] \arrow[r, "\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)"] & \mathrm{id}_{\mathcal{T}}[1]. \arrow[u, "g"']
\end{tikzcd}$$

In the language of Section 2.1.4, the class $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ is obtained by taking the Ext-class associated to the extension with splicing sequence

$$\mathrm{id}_{\mathcal{T}} \stackrel{\delta_1 G g}{\to} KG \stackrel{\alpha G}{\to} QG \stackrel{g^{-1}\delta_2 G}{\to} \mathrm{id}_{\mathcal{T}}\,.$$

Lemma 4.1.2. *If $\mathcal{T}_\varepsilon$ is a categorical deformation of a triangulated category $\mathcal{T}$, then there is a triangle*

$$E[-1] \stackrel{\iota}{\to} E[1] \stackrel{\sigma}{\to} KGE[1] \stackrel{\pi}{\to} E \tag{4.2}$$

in $D({\mathcal{T}_\varepsilon}^{\mathrm{op}} \otimes \mathcal{T})$. Moreover, consider the diagram

$$\begin{tikzcd}
 & KGE[1] \arrow[dr, "\pi"] \arrow[ddl, dashed] & \\
E[1] \arrow[ur, "\sigma"] \arrow[d, dashed, "\delta_1"'] & & E \arrow[ll, dashed, "\iota"'] \arrow[ddl, dashed, "\zeta_2"'] \\
K \arrow[rr, "\alpha"] \arrow[dr, "\zeta_1"'] & & Q \arrow[u, "\delta_2"'] \arrow[uul] \\
 & C \arrow[uul, "\eta_1"] \arrow[ur, "\eta_2"'] &
\end{tikzcd}$$

where the dotted arrows are of degree 1*. Then σ and π form commutative triangles with the faces, other than* (4.2)*, that contain them.*

Proof. This is the octahedral axiom for triangulated categories. $\square$

Composing with G and using $EG \cong \mathrm{id}_{\mathcal{T}}$, we obtain:

Corollary 4.1.3. *There is a triangle*

$$\mathrm{id}_{\mathcal{T}}[-1] \stackrel{\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)}{\to} \mathrm{id}_{\mathcal{T}}[1] \to KG[1] \to \mathrm{id}_{\mathcal{T}} \tag{4.3}$$

in $D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$.

4.1.2 Equivalences of deformations

Let $\mathcal{T}_\varepsilon, \tilde{\mathcal{T}}_\varepsilon$ be deformations of a triangulated category $\mathcal{T}$. Consider a quasi-functor

$$F\colon \mathcal{T}_\varepsilon \to \tilde{\mathcal{T}}_\varepsilon.$$

We will say that F is compatible with the semiorthogonal decompositions if the diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[d, "\mathrm{id}_{\mathcal{T}}"'] & \mathcal{T}_\varepsilon \arrow[l, "Q"'] \arrow[r, "E"] \arrow[d, "F"'] & \mathcal{T} \arrow[d, "\mathrm{id}_{\mathcal{T}}"'] \\
\mathcal{T} & \tilde{\mathcal{T}}_\varepsilon \arrow[l, "\tilde{Q}"'] \arrow[r, "\tilde{E}"] & \mathcal{T}
\end{tikzcd}$$

is 2-commutative, i.e. if there exist two natural isomorphisms

$$\chi_E\colon E \to \tilde{E}F \text{ and } \chi_Q\colon Q \to \tilde{Q}F. \tag{4.4}$$

Proposition 4.1.4. *The following are equivalent:*

- *The functor F is compatible with the semiorthogonal decompositions;*
- *There exists a triangle*
$$\tilde{G}E \to F \to \tilde{i}Q \to \tilde{G}E[1]. \tag{4.5}$$
in $D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \tilde{\mathcal{T}}_\varepsilon)$.

Proof. Assume that isomorphisms as in (4.4) are given. Since $\tilde{\mathcal{T}}_\varepsilon$ is a categorical extension, there exists a triangle
$$\tilde{G}\tilde{E} \to \mathrm{id}_{\tilde{\mathcal{T}}_\varepsilon} \to \tilde{i}\tilde{Q} \to \tilde{G}\tilde{E}[1];$$
composing on the right with F yields a triangle

$$\tilde{G}\tilde{E}F \to F \to \tilde{i}\tilde{Q}F \to \tilde{G}\tilde{E}F[1];$$

under the isomorphisms (4.4), this yields the triangle (4.5). Vice versa, assume given a triangle as in (4.5). Composing $\tilde{G}E \to F$ on the left with $\tilde{E}$ and using that $\tilde{E}\tilde{G} \cong \mathrm{id}_{\mathcal{T}}$, we obtain a map $E \to \tilde{E}F$ whose cone is given by $\tilde{E}\tilde{i}Q$; since $\tilde{E}\tilde{i} \cong 0$, this is an isomorphism. In the same way, one can compose on the left with $\tilde{Q}$ to obtain a natural isomorphism $Q \to \tilde{Q}F$.

□

Using the existence of the triangle (4.5) and reasoning as in the proof above, one also proves the following:

Corollary 4.1.5. *If F is compatible with the semiorthogonal decompositions, then there exist further natural isomorphisms*

$$\chi_G\colon \tilde{G} \to FG \text{ and } \chi_i\colon \tilde{i} \to Fi.$$

We will need the following technical fact:

Lemma 4.1.6. *The diagram*

$$\begin{tikzcd} & EG \arrow[d, "\chi_E G"] \\ \mathrm{id}_{\mathcal{T}} \arrow[ur, "g"] \arrow[dr, "\tilde{g}"'] & \tilde{E}FG \\ & \tilde{E}\tilde{G} \arrow[u, "\tilde{E}\chi_G"'] \end{tikzcd}$$

commutes.

Proof. Follows from writing down the various definitions. □

Recall that the deformation $\mathcal{T}_\varepsilon$ comes equipped with a class $\iota \in \mathrm{Ext}^2_{\mathrm{Fun}(\mathcal{T}_\varepsilon,\mathcal{T})}(E,E)$; similarly, $\tilde{\mathcal{T}}_\varepsilon$ comes with a class $\tilde{\iota} \in \mathrm{Ext}^2_{\mathrm{Fun}(\tilde{\mathcal{T}}_\varepsilon,\mathcal{T})}(\tilde{E},\tilde{E})$. Composing on the right with F determines a natural map

$$\mathrm{Ext}^2_{\mathrm{Fun}(\tilde{\mathcal{T}}_\varepsilon,\mathcal{T})}(\tilde{E},\tilde{E}) \to \mathrm{Ext}^2_{\mathrm{Fun}(\mathcal{T}_\varepsilon,\mathcal{T})}(\tilde{E}F,\tilde{E}F)$$

while the natural isomorphism χ_E gives an isomorphism

$$\mathrm{Ext}^2_{\mathrm{Fun}(T_\varepsilon,\mathcal{T})}(\tilde{E}F,\tilde{E}F) \cong \mathrm{Ext}^2_{\mathrm{Fun}(T_\varepsilon,\mathcal{T})}(I,I).$$

We will say that a quasi-equivalence F (compatible with the SODs) is an *equivalence of deformations* if the class $\tilde{\iota}$ corresponds to ι under these maps, i.e. if the diagram

$$\begin{tikzcd} E[-1] \arrow[r, "\iota"] \arrow[d, "{\chi_E[-1]}"'] & E[1] \arrow[d, "{\chi_E[1]}"] \\ \tilde{E}F[-1] \arrow[r, "\tilde{\iota}F"] & \tilde{E}F[1] \end{tikzcd} \tag{4.6}$$

commutes.

Remark. It is not strictly necessary to assume F to be a quasi-equivalence; however, it is always true that if a functor exists that is compatible with the SODs and that preserves the Ext-class, then there exists a quasi-equivalence with the same properties.

Proposition 4.1.7. *If there exists an equivalence between two deformations $\mathcal{T}_\varepsilon$ and $\tilde{\mathcal{T}}_\varepsilon$, then $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon) = \mu_{\mathcal{T}}(\tilde{\mathcal{T}}_\varepsilon)$*

Proof. Let F be such an equivalence. By definition, the diagram (4.6) commutes, hence the diagram

$$\begin{tikzcd} EG[-1] \arrow[r, "\iota G"] \arrow[d, "{\chi_E G[-1]}"'] & EG[1] \arrow[d, "{\chi_E G[1]}"] \\ \tilde{E}FG[-1] \arrow[r, "\tilde{\iota}FG"] & \tilde{E}FG[1] \end{tikzcd}$$

commutes as well. On the other hand, the diagram

$$\begin{tikzcd} \tilde{E}\tilde{G}[-1] \arrow[r, "\tilde{\iota}\tilde{G}"] \arrow[d, "{\tilde{E}\chi_G[-1]}"'] & \tilde{E}\tilde{G}[1] \arrow[d, "{\tilde{E}\chi_G[1]}"] \\ \tilde{E}FG[-1] \arrow[r, "\tilde{\iota}FG"] & \tilde{E}FG[1] \end{tikzcd}$$

commutes by naturality of $\tilde{\iota}$. Pasting these and using Lemma 4.1.6, we know that the diagram

$$\begin{tikzcd}
 & EG[-1] \arrow[r, "\iota G"] \arrow[d, "\chi_E G[-1]"] & EG[1] \arrow[d, "\chi_E G[1]"'] & \\
\mathrm{id}_{\mathcal{T}}[-1] \arrow[ur, "g"] \arrow[r] \arrow[dr, "\tilde{g}"'] & \tilde{E}FG[-1] \arrow[r, "\tilde{\iota}FG"] & \tilde{E}FG[1] & \mathrm{id}_{\mathcal{T}}[1] \arrow[ul, "g"'] \arrow[l] \arrow[dl, "\tilde{g}"] \\
 & \tilde{E}\tilde{G}[-1] \arrow[r, "\tilde{\iota}\tilde{G}"] \arrow[u, "\tilde{E}\chi_G[-1]"'] & \tilde{E}\tilde{G}[1] \arrow[u, "\tilde{E}\chi_G[1]"] &
\end{tikzcd}$$

commutes, i.e. $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon) = \mu_{\mathcal{T}}(\tilde{\mathcal{T}_\varepsilon})$. □

Definition 4.1.8. Define the set $\mathrm{CatDef}_{\mathcal{T}}(k[\varepsilon])$ as the set of categorical deformations of $\mathcal{T}$ up to equivalence of deformations.

By Proposition 4.1.7, the map that assigns to a categorical deformation $\mathcal{T}_\varepsilon$ the class $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ descends to the quotient, defining a morphism

$$\mu_{\mathcal{T}}\colon\ \mathrm{CatDef}_{\mathcal{T}}(k[\varepsilon]) \to \mathrm{HH}^2(\mathcal{T}).$$

Our first main result is the following

Theorem 4.1.9. *The map $\mu_{\mathcal{T}}$ defines a bijection between $\mathrm{HH}^2(\mathcal{T})$ and the set $\mathrm{CatDef}_{\mathcal{T}}(k[\varepsilon])$ of equivalence classes of categorical deformations of $\mathcal{T}$.*

To prove Theorem 4.1.9, we construct an explicit inverse to $\mu_{\mathcal{T}}$ in §4.1.3. The proof will be completed in §4.2.

4.1.3 Constructing an inverse

4.1.3.1 The construction

Consider a class $\mu \in \mathrm{HH}^2(\mathcal{T})$; we see μ as a natural transformation $\mathrm{id}_{\mathcal{T}}[-1] \xrightarrow{\mu} \mathrm{id}_{\mathcal{T}}[1]$, and complete it (non-canonically) to a triangle

$$\mathrm{id}_{\mathcal{T}}[-1] \xrightarrow{\mu} \mathrm{id}_{\mathcal{T}}[1] \xrightarrow{\nu} \phi \xrightarrow{\omega} \mathrm{id}_{\mathcal{T}}$$

in $D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$. By construction ϕ is a quasi-functor $\mathcal{T} \to \mathcal{T}$ and we can construct the gluing

$$\mathcal{T}_\varepsilon = \mathcal{T} \times_\phi \mathcal{T};$$

this comes equipped with the various functors and natural transformations described in 2.3.4. To construct the Yoneda extension, begin by defining the natural transformation $E \xrightarrow{\delta_1} K$ as the composition

$$E \xrightarrow{\nu E} \phi E[-1] \xrightarrow{\gamma} K$$

where γ comes from (2.8). Complete now δ_1 to a triangle

$$E \xrightarrow{\delta_1} K \xrightarrow{\zeta_1} C \xrightarrow{\eta_1} E[1].$$

The octahedral axiom proves:

Lemma 4.1.10. *There is a triangle*

$$C \xrightarrow{\eta_2} Q \xrightarrow{\delta_2} E \xrightarrow{\zeta_2} C[1]. \tag{4.7}$$

in $D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \mathcal{T})$. Moreover, consider the diagram

$$\begin{tikzcd}
 & C \arrow[dr, "\eta_2"] \arrow[ddl, dashed, "\eta_1"'] & \\
E[-1] \arrow[ur, "\zeta_2"] \arrow[d, dashed, "\mu E"'] & & Q \arrow[ll, dashed, "\delta_2"'] \arrow[ddl, dashed] \\
E \arrow[rr, "\delta_1"] \arrow[dr, "\nu E"'] & & K \arrow[u, "\alpha"'] \arrow[uul, "\zeta_1"] \\
 & \phi E[-1] \arrow[ur, "\gamma"'] \arrow[uul] &
\end{tikzcd}$$

where the dotted arrows have degree 1*. Then ζ_2 and η_2 form commutative triangles with the faces other than* (4.7) *that contain them.*

Hence, the composition $K \xrightarrow{\zeta_1} C \xrightarrow{\eta_2} Q$ equals α and the two triangles

$$E \xrightarrow{\delta_1} K \xrightarrow{\zeta_1} C \xrightarrow{\eta_1} E[1] \text{ and } C \xrightarrow{\eta_2} Q \xrightarrow{\delta_2} E \xrightarrow{\zeta_2} C[1]$$

give a Yoneda 2-extension compatible with the semiorthogonal decomposition. We have thus proved:

Proposition 4.1.11. *For any Hochschild class $\mu \in \mathrm{HH}^2(\mathcal{T})$, the category $\mathcal{T}_\varepsilon$ constructed above is a categorical deformation of $\mathcal{T}$.*

4.1.3.2 Well-definedness

In our construction we made two arbitrary choices: the choices of the cones ϕ of μ and C of δ_1; we will show that in each casedifferent choices give rise to equivalent deformations. For δ_1, this is very easy; since different choices of cones do not alter the underlying category, we can pick the identity functor as an equivalence; this clearly preserves the class ι, since that is not altered by a different choice of cone for δ_1. On the other hand, changing the choice of cone of μ does change the underlying category; let ϕ' be another choice for that cone. Then we always have a (noncanonical) isomorphism $f\colon \phi \to \phi'$ in $D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$. Assume for now that this f is given by a honest map of dg bimodules; then one can define the functor

$$\begin{gathered}
\mathcal{T}_\varepsilon \times_\phi \mathcal{T}_\varepsilon \to \mathcal{T}_\varepsilon \times_{\phi'} \mathcal{T}_\varepsilon \\
(M_1, M_2, n) \to (M_1, M_2, f(n))
\end{gathered}$$

which, for the same reason as above, is readily seen to be an equivalence of deformations. In the case where f is instead given by a zigzag of quasi-isomorphism, we simply obtain a zigzag of quasi-functors.

4.2 Proof of Theorem 4.1.9

In the previous section, we have shown that there is a well-defined map

$$\mathrm{HH}^2(\mathcal{T}) \to \mathrm{CatDef}_{\mathcal{T}}(k[\varepsilon])$$

which assigns to a Hochschild class the gluing along its cone. In this section we show that this gives an inverse of $\mu_{\mathcal{T}}$.

4.2.1 Class to class

First we have to show that if we start with a class μ, take its cone ϕ, construct the gluing $\mathcal{T}_\varepsilon = \mathcal{T} \times_\phi \mathcal{T}$ and then take the class $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ we recover the starting class μ. This essentially follows from Lemma 4.1.10: by construction, the class $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ is obtained by first taking the composition

$$E[-1] \overset{\zeta_2}{\to} C \overset{\eta_1}{\to} E[1]$$

and then composing it with G. By Lemma 4.1.10 we know that $\eta_1\zeta_2 = \mu E$, so we only have to show that the diagram

$$\begin{tikzcd} \mathrm{id}_{\mathcal{T}}[-1] \arrow[r, "\mu"] \arrow[d, "g"] & \mathrm{id}_{\mathcal{T}}[1] \arrow[d, "g"'] \\ EG[-1] \arrow[r, "\mu EG"] & EG[1] \end{tikzcd}$$

commutes; but that follows by naturality of μ.

4.2.2 Deformation to deformation

We now have to prove that if we start with a deformation $\mathcal{T}_\varepsilon$, take the associated class $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$, take its cone ϕ, then the gluing $\mathcal{T}'_\varepsilon = \mathcal{T} \times_\phi \mathcal{T}$ is equivalent to $\mathcal{T}_\varepsilon$ as a deformation.

We first ought to construct a suitable quasi-functor $F \colon \mathcal{T}_\varepsilon \to \mathcal{T} \times_\phi \mathcal{T}$; we have the triangle

$$\mathrm{id}_{\mathcal{T}}[-1] \overset{\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)}{\to} \mathrm{id}_{\mathcal{T}}[1] \to KG[1] \to \mathrm{id}_{\mathcal{T}};$$

from Corollary 4.1.3 and, since choosing a different cone for $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ yields equivalent deformations (see Section 4.1.3.2) we can pick $\phi = KG[1]$. Denoting with $\tilde{i}, \tilde{G}, \tilde{K}$ the relevant functors of the gluing $\mathcal{T} \times_{KG[1]} \mathcal{T}$, observe that $\tilde{K}\tilde{i} = KG$ by construction – the composition of the two adjoints always equals the gluing functor. Recall that, by virtue of being an extension, the category $\mathcal{T}_\varepsilon$ comes equipped with a morphism

$$iQ \to GE[1]$$

in $D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \mathcal{T}_\varepsilon)$. We then have

$$\begin{aligned} \mathrm{Hom}_{D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \mathcal{T}_\varepsilon)}(iQ, GE[1]) &\cong \mathrm{Hom}_{D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \mathcal{T})}(Q, KGE[1]) = \mathrm{Hom}_{D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes \mathcal{T})}(Q, \tilde{K}\tilde{G}E[1]) \\ &\cong \mathrm{Hom}_{D(\mathcal{T}_\varepsilon{}^{\mathrm{op}} \otimes (\mathcal{T} \times_{KG} \mathcal{T}))}(\tilde{i}Q, \tilde{G}E[1]) \end{aligned}$$

whence we obtain a morphism $\tilde{i}Q \to \tilde{G}E[1]$ that we can complete to a triangle

$$\tilde{G}E \to F \to \tilde{i}Q \to \tilde{G}E[1].$$

By the same argument as [KL12, Proposition 7.7] the bimodule F is a quasi-functor. To see that F is a quasi-equivalence, the same process can be applied to obtain a triangle

$$G\tilde{E} \to F' \to i\tilde{Q} \to G\tilde{E}[1]$$

and one sees that F' is a quasi-inverse to F.

Remark. Heuristically, the functor F is given by

$$\begin{aligned}&\mathcal{T}_\varepsilon \to \mathcal{T} \times_{KG[1]} \mathcal{T}\\ &M \to (QM, EM, QM \to KGEM[1]),\end{aligned}$$

where the map $QM \to KGEM[1]$ is obtained from the canonical map $iQ \to GE[1]$ via the isomorphism

$$\mathrm{Hom}(iQ, GE[1]) \cong \mathrm{Hom}(Q, KGE[1]).$$

One can then also directly show along the lines of [KL12, Proposition 4.14] that F is an equivalence.

To conclude, we will need the following lemma.

Lemma 4.2.1. *Let $\mathcal{T}_\varepsilon$ be a categorical deformation. Then the induced natural transformation $\iota\colon E[-1] \to E[1]$ coincides with $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)E$.*

Proof. Consider the natural isomorphism $E \overset{gE}{\to} EGE$. The diagram

$$\begin{tikzcd} E[-1] \arrow[r, "gE"] \arrow[d, "\iota"] & EGE[-1] \arrow[d, "\iota GE"'] \\ E[1] \arrow[r, "gE"] & EGE[1] \end{tikzcd}$$

commutes by naturality of ι. The natural transformation $\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)$ is defined via the square

$$\begin{tikzcd} \mathrm{id}_{\mathcal{T}}[-1] \arrow[r, "g"] \arrow[d, "\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)"] & EG[-1] \arrow[d, "\iota G"'] \\ \mathrm{id}_{\mathcal{T}}[1] \arrow[r, "g"] & EG[1] \end{tikzcd}$$

hence the diagram

$$\begin{tikzcd} E[-1] \arrow[r, "gE"] \arrow[d, "\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)E"] & EGE[-1] \arrow[d, "\iota GE"] \\ E[1] \arrow[r, "gE"] & EGE[1] \end{tikzcd}$$

must commute. Pasting these we obtain a commutative diagram

$$\begin{tikzcd} E[-1] \arrow[r, "gE"] \arrow[d, "\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)E."] & EGE[-1] \arrow[d, "\iota GE"] & E[-1] \arrow[l, "gE"'] \arrow[d, "\iota"] \\ E[1] \arrow[r, "gE"] & EGE[1] & E[1] \arrow[l, "gE"'] \end{tikzcd}$$

Since the horizontal arrows are isomorphisms, we get the claim. $\square$

Remark. For the proof we never used that $\mathcal{T}_\varepsilon$ was a deformation, but only that it is an extension. Indeed the same proof shows that any natural transformation $E \to E$ can be recovered from the induced map $\mathrm{id}_\mathcal{T} \cong EG \to EG \cong \mathrm{id}_\mathcal{T}$.

We can now prove that F is an equivalence of deformations. The goal is to show that the diagram

$$\begin{tikzcd} EF[-1] \arrow[r, "\iota F"] \arrow[d, "\chi_E"] & EF[1] \arrow[d, "\chi_E"] \\ E'[-1] \arrow[r, "\iota'"] & E'[1] \end{tikzcd}$$

is commutative. Using Lemma 4.2.1 and that by definition $\iota' = \mu_\mathcal{T}(\mathcal{T}_\varepsilon)E'$, this is the same diagram as

$$\begin{tikzcd} EF[-1] \arrow[r, "\mu(\mathcal{T}_\varepsilon)EF"] \arrow[d, "\chi_E"] & EF[1] \arrow[d, "\chi_E"] \\ E'[-1] \arrow[r, "\mu(\mathcal{T}_\varepsilon)E'"] & E'[1] \end{tikzcd}$$

and now the claim follows by naturality of $\mu_\mathcal{T}(T_\varepsilon)$.

Chapter 5

The 1-derived category as a deformation

We have arrived at the last chapter of the thesis. We are left with two tasks: showing that the 1-derived category is indeed a deformation of $D(A)$, and finding an appropriate version of the square (1.1). The first is simple enough, and is done in Section 5.1.1. In Section 5.1.2 we compute explicitly the Hochschild class identifies by seeing the 1-derived category as a deformation (Theorem 5.1.1). In Section 5.2 we define a set of *curved Morita deformations*, which are a version of the objects considered in [KL09] which allow for cdg – and not just dg – algebra deformations. We show (Theorem 5.2.1) that these deformations are, as one would expect, parametrized by the second Hochschild cohomology. In Section 5.3, we pull all the strings together: using the notions of curved Morita deformations and categorical deformations, we recover the derived equivalent of the square (1.1). After that, in Section 5.4.1 we show how categorical deformations can be thought as "blowups" of classical deformations, and describe explicitly the case of an algebra deformation. Finally, Section 5.5 contains some technical results about extensions of A_∞ functors which are needed in the proof of Theorem 5.1.1.

5.1 The 1-derived category

Let A be a dg k-algebra. As in the introduction, we will denote with A_ε a first order algebra deformation of A, and with $D^\varepsilon(A_\varepsilon)$ its 1-derived category $D^1(A_1)$.The goal of this section is to prove the following:

Theorem 5.1.1. *Let A be a dg algebra, and A_ε a cdg deformation of A corresponding to a Hochschild cocycle $\mu_A \in \mathbf{C}^2(A)$. Then the category $D^\varepsilon(A_\varepsilon)$ is a categorical deformation of $D(A)$, and the class $\mu_{D(A)}(D^\varepsilon(A_\varepsilon)) \in \mathrm{HH}^2(D(A))$ coincides with $\chi_A(\mu_A)$.*

Note that we have used the notation μ_A to denote both the cocycle in $\mathbf{C}^2(A)$ and the corresponding class in $\mathrm{HH}^2(A)$; this makes sense because the map χ_A is defined at the level of cocycles, and not only for classes.

Remark. Specifically, the above result implies (in fact, since we are operating at the level of classes is equivalent to) the fact that the image of $\mu_{D(A)}(D^\varepsilon(A_\varepsilon))$ via the restriction $\mathrm{HH}^2(D(A)) \to \mathrm{HH}^2(A)$ coincides with μ_A.

5.1.1 Categorical deformations and the 1-derived category

It is straightforward to see that $D^{\varepsilon}(A_{\varepsilon})$ is a categorical deformation of $D(A)$. We know from Theorem 3.5.7 that there is a recollement

$$\begin{tikzcd} D(A) \arrow[r, hook, "i"] & D^{\varepsilon}(A_{\varepsilon}) \arrow[l, bend right, "\operatorname{Coker} t"'] \arrow[l, bend left, "\operatorname{Ker} t"] \arrow[r, "\operatorname{Im} t"] & D(A). \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

For concreteness, we will consider the dg enhancement of $D^{\varepsilon}(A_{\varepsilon})$ given by the A_{ε}-modules which are cofibrant according to the model structure from Section 3.7; with this enhancement, the functors i, $\operatorname{Coker} t$, $\operatorname{Ker} t$, $\operatorname{Im} t$ are represented by honest dg functors while G is given by a quasi-functor. The recollement induces a natural transformation $\operatorname{Ker} t \overset{\alpha}{\to} \operatorname{Coker} t$ which projects the subobject $\operatorname{Ker} t_M \subseteq M$ into the quotient $\operatorname{Coker} t_M \cong M/tM$. Moreover, there are natural transformations $\operatorname{Im} t \overset{\delta_1}{\to} \operatorname{Ker} t$ and $\operatorname{Coker} t \overset{\delta_2}{\to} \operatorname{Im} t$ induced respectively by the inclusion $tM \hookrightarrow \operatorname{Ker} t_M$ and the multiplication $M/tM \overset{t}{\to} tM$. For any A_{ε}-module M, the sequence

$$0 \to tM \overset{\delta_1}{\to} \operatorname{Ker} t_M \overset{\alpha}{\to} \operatorname{Coker} t_M \overset{\delta_2}{\to} \operatorname{Im} tM \to 0$$

is exact. Splitting this yields short exact sequences

$$0 \to tM \overset{\delta_1}{\to} \operatorname{Ker} t_M \to \frac{\operatorname{Ker} t_M}{tM} \to 0 \text{ and } 0 \to \frac{\operatorname{Ker} t_M}{tM} \to \operatorname{Coker} t_M \overset{\delta_2}{\to} tM \to 0$$

which are natural in M, and have the property that the composition

$$\operatorname{Ker} t_M \to \frac{\operatorname{Ker} t_M}{tM} \to \operatorname{Coker} t_M$$

coincides with α. Since semifree A-modules are in particular projective as graded A-modules, the functor $\operatorname{Hom}_A(N, -)$ is exact and we have two short exact sequences of $D^{\varepsilon}(A_{\varepsilon})$-$D(A)$ bimodules

$$0 \to \operatorname{Hom}_A(-, \operatorname{Im} t) \overset{\delta_1}{\to} \operatorname{Hom}_A(-, \operatorname{Ker} t) \to \operatorname{Hom}_A(-, \frac{\operatorname{Ker} t}{\operatorname{Im} t}) \to 0$$

and

$$0 \to \operatorname{Hom}_A(-, \frac{\operatorname{Ker} t}{\operatorname{Im} t}) \to \operatorname{Hom}_A(-, \operatorname{Coker} t) \overset{\delta_2}{\to} \operatorname{Hom}_A(-, \operatorname{Im} t) \to 0$$

with the property that the composition

$$\operatorname{Hom}_A(-, \operatorname{Ker} t) \to \operatorname{Hom}_A(-, \frac{\operatorname{Ker} t}{\operatorname{Im} t}) \to \operatorname{Hom}_A(-, \operatorname{Coker} t)$$

coincides with α. Since short exact sequences give triangles in the derived category, we are done. Note that the role of the functor C from Definition 4.1.1 is taken by the dg functor $\frac{\operatorname{Ker} t}{\operatorname{Im} t}$.

Remark. It follows from Proposition 3.6.10 that the subcategory $D^{\mathrm{si}}(A_\varepsilon) \subseteq D^\varepsilon(A_\varepsilon)$ given by the semiderived category ([Pos18]) coincides with the kernel of the functor C – the cone of the natural transformation δ_1. This observation allows to define, for any categorical deformation $\mathcal{T}_\varepsilon$, a full subcategory $\mathcal{T}^{\mathrm{si}} \subseteq \mathcal{T}_\varepsilon$ given by the kernel of the functor C appearing in the deformation data. This subcategory measures – at least in the compactly generated case – how close the deformation is to admit a classical, or uncurved, representative. Specifically, in the case of an algebra deformation, it was already observed in the discussion after Proposition 3.6.10 that as soon an object $M \in D^{\mathrm{si}}(A_\varepsilon)$ exists whose reduction M/tM is a compact generator of $D(A)$, the deformation A_ε is equivalent to an uncurved one. We expect a similar behavior to also appear in the general case.

On the other hand, we were not able to find any characterization for the quotient $D(A_\varepsilon)$ of $D^\varepsilon(A_\varepsilon)$, a priori only defined in the case where A_ε is uncurved, which only makes use of the deformation data. This remains an open question; for some relevant discussion, see Section 5.4.

Example 5.1.1. In the case of the graded field considered in Example 3.2.2, the semiorthogonal decomposition of $D^\varepsilon(A_\varepsilon)$ is actually orthogonal. This is consistent with our theory: since the Hochschild cocycle u is an isomorphism, the same holds for the cocycle in $\chi_A(u) \in Z^2\mathbf{C}(D(A))$. Hence, its cone – the gluing functor – is the zero object.

5.1.2 Enhancing the left adjoint

As mentioned before, the functor G is not induced by a dg functor. It follows essentially by definition that it does have an enhancement as a quasi-functor, but this is impractical: concretely, the natural formula must be resolved in order to do concrete computations and this creates complications. Instead, we will see that G has a very natural incarnation as an A_∞-functor. We refer the reader to [COS19; COS24] for details regarding the switch between different models for the homotopy category of dg categories. Note that we use for A_∞-functors the same notion of adjunction we used for quasi-functors; this is also equivalent (at the homotopy level) to the notion of adjunction between ∞-functors, see [DKSS24].

We will consider a variant of the construction from Section 3.2 of one of the compact generators of the 1-derived category. Recall that the curvature of A_ε is of the form $t\mu_0$. We have a well-defined degree 2 closed A-module map $A \overset{t\mu_0}{\to} A_\varepsilon$. We define the A_ε-module Γ as the "two sided cone" of the diagram

$$A \underset{\pi}{\overset{-t\mu_0}{\rightleftarrows}} A_\varepsilon$$

where π is the natural projection. Explicitly, Γ is given by the graded module $A_\varepsilon \oplus A[-1]$ with differential given by

$$d_\Gamma(a_\varepsilon, b) = (d_{A_\varepsilon} a_\varepsilon - t\mu_0 b, d_{A[-1]} b + a),$$

where we have denoted the action of the map π by removing the subscript ε.

Proposition 5.1.2. *The module Γ is a cdg A_ε-module.*

Proof. Straightforward, see the proof of Proposition 3.2.2. □

Since $t\Gamma \cong A$, there is a natural dg algebra map

$$E = \mathrm{Hom}_{A_\varepsilon}(\Gamma, \Gamma) \to \mathrm{Hom}_A(t\Gamma, t\Gamma) \cong A$$

which is easily seen to be a surjective quasi-isomorphism (by the same argument used in the proof of Lemma 5.5). Hence, it admits an A_∞-inverse $e\colon A \to E$. Let us describe explicitly this inverse, remembering that the A_∞-algebra morphism e is given by a collection

$$g_i \in \mathrm{Hom}_k(A^{\otimes i}, E)$$

of degree $1-i$ satisfying the A_∞-identities

$$\sum_{n=r+s+t} (-1)^{r+st} g_u(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes r}) = \sum_{i_1+\ldots i_r = n} (-1)^s m_r(g_{i_1} \otimes g_{i_2} \otimes \ldots \otimes g_{i_r})$$

where m_i represent the operations of A and E, and $s = \sum_j (r-j)(i_j - 1)$. At the graded level, the algebra E is the matrix algebra

$$\begin{bmatrix} \mathrm{Hom}_{A_\varepsilon}(A_\varepsilon, A_\varepsilon) & \mathrm{Hom}_{A_\varepsilon}(A[-1], A_\varepsilon) \\ \mathrm{Hom}_{A_\varepsilon}(A_\varepsilon, A[-1]) & \mathrm{Hom}_{A_\varepsilon}(A, A) \end{bmatrix} \cong \begin{bmatrix} A_\varepsilon & A[1] \\ A[-1] & A \end{bmatrix}$$

with differential

$$d_E \begin{bmatrix} x_\varepsilon & y \\ z & w \end{bmatrix} = \begin{bmatrix} d_\varepsilon x_\varepsilon & dy \\ dz & dw \end{bmatrix} + \begin{bmatrix} -t(y + \mu_0 z) & \mu_0 w - x\mu_0 \\ x - w & 0 \end{bmatrix}$$

for degree 0 elements x_ε, y, z, w – otherwise the Koszul sign rule applies. The component g_1 carries the element $a \in A$ to the matrix

$$\begin{bmatrix} a & \mu_1(A) \\ 0 & a \end{bmatrix};$$

this does commute with the differentials, but not with the multiplications – since the map $A \overset{1\to 1}{\to} A_\varepsilon$ is not a map of associative algebras, neither is g_1. A higher component $g_2\colon A \otimes_k A \to E$ is therefore needed. We define

$$g_2(a, b) = \begin{bmatrix} 0 & \mu_2(a, b) \\ 0 & 0 \end{bmatrix}.$$

Proposition 5.1.3. *The map $A \overset{g}{\to} E$ is a quasi-isomorphism of A_∞-algebras.*

Proof. By definition, g is a quasi-isomorphism if and only if g_1 is; but g_1 is a right inverse to the quasi-isomorphism $E \to A$, so the only thing to check is that g is indeed an A_∞-algebra morphism. Since A and E are dg algebras and $g_i = 0$ for $i > 2$, the A_∞ equations specialize to

$$d_E g_1(a) = g_1 d_A(a) \tag{5.1}$$
$$g_1(ab) - g_1(a)g_1(b) = d_E g_2(a, b) + g_2(d_A a, b) + g_2(a, d_A b) \tag{5.2}$$
$$g_2(a, b)g_1(c) - g_1(a)g_2(a, b) + g_2(ab, c) - g_2(a, bc) = 0 \tag{5.3}$$
$$g_2(a, b)g_2(b, c) = 0 \tag{5.4}$$

for degree 0 elements $a, b, c \in A$ – otherwise the Koszul sign rule must be applied. Writing down the explicit formulas, one sees that equation (5.1) corresponds to the condition $d\mu_1 + \mu_1 d = [\mu_0, -]$, equation (5.2) to

$$a\mu_1(b) - \mu_1(ab) + \mu_1(a)b = d\mu_2(a, b) - \mu_2(a, db) - \mu_2(a, db)$$

and equation (5.3) to

$$a\mu_2(b, c) - \mu_2(ab, c) + \mu_2(a, bc) - \mu_2(a, b)c.$$

Equation (5.4) is automatically satisfied, since g_2 is a square-zero matrix. There are precisely (all but one of) the components of the equation $d_H(\mu) = 0$, where d_H is the total Hochschild differential. Note how the only condition that was not needed in this proof – that is, $d_A\mu_0 = 0$ – was implicitly used in showing that Γ is a cdg A_ε-module. □

Then, seeing A as a one-object dg category, we have a well-defined A_∞-functor

$$A \xrightarrow{g} D^\varepsilon(A_\varepsilon)$$

which sends the only object A to $\Gamma \in D^\varepsilon(A_\varepsilon)$.

5.1.2.1 Cones of A_∞-morphisms

Let $F, G\colon A \to D(A)$ be (strict) A_∞-functors and $\eta\colon F \to G$ an A_∞-natural transformation. In components, F is given by a certain object $FA \in D(A)$ together with an A_∞-algebra map

$$A \xrightarrow{F} \mathrm{Hom}_A(FA, FA)$$

which is given explicitly by components

$$F_i\colon A^{\otimes i} \to \mathrm{Hom}_A(FA, FA)$$

of appropriate degree, and the same for G. Similarly, the natural transformation η is given by components

$$\eta_i\colon A^{\otimes i} \to \mathrm{Hom}_A(FA, GA).$$

The category of A_∞-functors $A \to D(A)$ is pretriangulated, and we can give an explicit description of cones in this category, following [Lef03]. At the object level, the A_∞-functor $\mathrm{Cone}(\eta)(A)$ is defined as

$$\mathrm{Cone}(\eta)(A) = \mathrm{Cone}(\eta_0) \cong_{gr} GA \oplus FA[1].$$

The A_∞-morphism

$$A \to \mathrm{Hom}_A(\mathrm{Cone}(\eta_0), \mathrm{Cone}(\eta_0)) \cong \mathrm{Hom}_A(GA \oplus FA[1], GA \oplus FA[1])$$

is given in components by

$$\begin{bmatrix} F_i & \eta_i \\ 0 & G_i \end{bmatrix} : A^{\otimes i} \to \mathrm{Hom}_A(GA \oplus FA[1], GA \oplus FA[1]).$$

Any such morphism defines a canonical short exact sequence of A_∞-functors

$$0 \to GA \to \mathrm{Cone}(\eta) \to FA[1] \to 0$$

which corresponds to the triangle

$$FA \xrightarrow{\eta} GA \to \mathrm{Cone}(\eta) \to FA[1]$$

in the homotopy category.

Lemma 5.1.4. *The composition*

$$A \xrightarrow{g} D^\varepsilon(A_\varepsilon) \xrightarrow{\mathrm{Ker}\, t} D(A)$$

is isomorphic to the cone of μ_A, when seen as a closed morphism $\mathrm{inc}[-2] \to \mathrm{inc}$*. Moreover the composition*

$$A \xrightarrow{g} D^\varepsilon(A_\varepsilon) \xrightarrow{\mathrm{Coker}\, t} D(A)$$

coincides with the cone of the identity natural transformation of $\mathrm{inc}[-1]$*.*

Proof. For the first statement one observes that, at the graded level, $\mathrm{Ker}\, t_\Gamma = A \oplus A[-1]$, with differential induced by Γ. Under the isomorphism $\mathrm{Ker}\, A_\varepsilon \cong A$, we see that

$$d_{\mathrm{Ker}\, t_\Gamma}(a, b) = (da, db + a\mu_0)$$

which shows that $\mathrm{Ker}\, t_\Gamma$ corresponds to the cone of μ_0. Hence, as an object of $D(A)$, $\mathrm{Ker}\, t_\Gamma$ coincides with the cone of μ. To conclude, we have to show that this also holds at the level of the actions, i.e. that the two possible A_∞-algebra maps

$$A \to \mathrm{Hom}_A(\mathrm{Ker}\, t_\Gamma, \mathrm{Ker}\, t_\Gamma)$$

– one defined by seeing $\mathrm{Ker}\, t_\Gamma$ as the cone of μ_A, the other induced by e – coincide. This is however immediate, since they are both given in components by

$$\begin{aligned} A &\to \mathrm{Hom}_A(\mathrm{Ker}\, t_\Gamma, \mathrm{Ker}\, t_\Gamma) \\ a &\to \begin{bmatrix} a & \mu_1(a) \\ 0 & a \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} A \otimes A &\to \mathrm{Hom}_A(\mathrm{Ker}\, t_\Gamma, \mathrm{Ker}\, t_\Gamma) \\ a \otimes b &\to \begin{bmatrix} 0 & \mu_2(a, b) \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

The second statement is straightforward using the same argument. □

5.1.2.2 The adjunction isomorphism

Using Proposition 5.5.1, we can extend the functor g to an A_∞-functor

$$G_\infty := \chi_A(g)\colon D(A) \to D^\varepsilon(A_\varepsilon).$$

Given $M = (\{n_i\}, f) \in \mathrm{Tw}(A)$, we have a natural isomorphism

$$tG_\infty M \cong (\oplus_i t\Gamma[n_i], g_1(f)_{|t\Gamma}) \cong (\oplus_i A[n_i], f) \cong M$$

We therefore have a natural isomorphism $\mathrm{id}_{D(A)} \to \mathrm{Im}\, t \circ G_\infty$. This induces a natural transformation

$$\mathrm{Hom}_{A_\varepsilon}(G_\infty M, N) \to \mathrm{Hom}_A(tG_\infty M, tN) \to \mathrm{Hom}_A(M, tN).$$

We can show that this emerges as the unit of an adjunction between $\mathrm{Im}\, t$ and G_∞.

Lemma 5.1.5. *The natural morphism*

$$\mathrm{Hom}_{A_\varepsilon}(G_\infty M, N) \cong \mathrm{Hom}_A(M, tN) \tag{5.5}$$

is an isomorphism in $D(k)$.

Proof. Consider the subcategory of $D(A)$ given by the modules for which (5.5) is an isomorphism. This is closed under shifts and cones, as well as under coproducts since G_∞ preserves them. Hence it is enough to show that it contains the generator A; since $G_\infty A = \Gamma$, we want to show that the map

$$\mathrm{Hom}_{A_\varepsilon}(\Gamma, M) \to \mathrm{Hom}_A(t\Gamma, tM) \cong \mathrm{Hom}_A(A, tM) \cong tM$$

is a quasi-isomorphism. It is immediate to check that it is surjective, and its kernel is given by

$$\mathrm{Hom}_{A_\varepsilon}(\Gamma, \mathrm{Ker}\, t_M) \cong \mathrm{Hom}_A(Q\Gamma, \mathrm{Ker}\, t_M)$$

which is acyclic since $Q\Gamma$ is contractible. □

Hence by [DKSS24, Proposition 1.2.7] the functor G_∞ is left adjoint to $\mathrm{Im}\, t$.

We will also need the "large" version of Lemma 5.1.4.

Lemma 5.1.6. *The composition*

$$D(A) \overset{G_\infty}{\to} D^\varepsilon(A_\varepsilon) \overset{\mathrm{Ker}\, t}{\to} D(A)$$

is isomorphic to the cone of $\chi_A(\mu_A)$, when seen as a closed morphism $\mathrm{id}_{D(A)}[-2] \to \mathrm{id}_{D(A)}$. Moreover the composition

$$D(A) \overset{G_\infty}{\to} D^\varepsilon(A_\varepsilon) \overset{\mathrm{Coker}\, t}{\to} D(A)$$

coincides with the cone of the identity natural transformation of $\mathrm{id}_{D(A)}[-1]$.

Proof. This is a combination of Lemma 5.1.4 and the properties of the functor χ_A. □

Proof of Theorem 5.1.1. First of all, observe that the unit map $\mathrm{Im}\, t \circ G_\infty \to \mathrm{id}_{D(A)}$ is the identity. Thus by definition, the class $\mu_{D(A)}(D^\varepsilon(A_\varepsilon))$ is obtained by composing with G_∞ the morphism

$$\mathrm{Im}\, t \to \mathrm{Im}\, t[2]$$

obtained by composing the boundary maps induced by the two short exact sequences

$$0 \to \frac{\operatorname{Ker} t}{\operatorname{Im} t} \to \operatorname{Coker} t \xrightarrow{\delta_2} \operatorname{Im} t \to 0$$

and

$$0 \to \operatorname{Im} t \xrightarrow{\delta_1} \operatorname{Ker} t \to \frac{\operatorname{Ker} t}{\operatorname{Im} t} \to 0.$$

The comparison between quasi-functors and A_∞-functors preserves dg functors and dg natural transformations between them, hence the class $\mu_{D(A)}(D^\varepsilon(A_\varepsilon))$ is given, in the A_∞ setting, by the composition of the morphism

$$\mathrm{id}_A \to \frac{\operatorname{Ker} t}{\operatorname{Im} t} \circ G_\infty[1]$$

induced by the short exact sequence

$$0 \to \frac{\operatorname{Ker} t}{\operatorname{Im} t} \circ G_\infty \to \operatorname{Coker} t \circ G_\infty \xrightarrow{\delta_2 G} \operatorname{Im} t \circ G \cong \mathrm{id}_A \to 0 \tag{5.6}$$

with the morphism

$$\frac{\operatorname{Ker} t}{\operatorname{Im} t} \circ G_\infty[1] \to \mathrm{id}_{D(A)}[2]$$

induced by the short exact sequence

$$0 \to \mathrm{id}_{D(A)} \cong \operatorname{Im} t \circ G_\infty \xrightarrow{\delta_1} \operatorname{Ker} t \circ G_\infty \to \frac{\operatorname{Ker} t}{\operatorname{Im} t} \circ G_\infty \to 0. \tag{5.7}$$

The first observation is that $\frac{\operatorname{Ker} t}{\operatorname{Im} t} \circ G_\infty$ coincides with $\mathrm{id}_{D(A)}[-1]$; moreover by Proposition 5.5.1 the short exact sequence (5.6) can be read as

$$0 \to \mathrm{id}_{D(A)}[-1] \to \operatorname{Cone}(\mathrm{id}_{\mathrm{id}_{D(A)}[-1]}) \to \mathrm{id}_{D(A)} \to 0$$

from which[1] one can conclude that the boundary map is the identity map $\mathrm{id}_{D(A)} \to \mathrm{id}_{D(A)}$. In the same way, the short exact sequence (5.7) reads

$$0 \to \mathrm{id}_{D(A)}[1] \to \operatorname{Cone}(\chi_A(\mu_A)) \to \mathrm{id}_{D(A)} \to 0$$

from which it follows that the boundary map is precisely $\chi_A(\mu_A)$, hence the claim.

□

5.2 Curved Morita deformations

In this section, we place one of the missing pieces: we show that, if one allows for curved deformations, then Morita deformations are indeed parametrized by the Hochschild complex.

Precisely, we prove:

[1]The confusing notation $\mathrm{id}_{\mathrm{id}_{D(A)}[-1]}$ denotes the identity natural transformation between the shifted identity functor and itself.

Theorem 5.2.1. *Let* $\mathrm{cDef}_A(k[\varepsilon])$ *denote the set of equivalence classes of curved Morita deformations of a dg algebra* A*. There exists a bijection*

$$\nu \colon \mathrm{cDef}_A(k[\varepsilon]) \to \mathrm{HH}^2(A).$$

This is not too surprising: the lack of surjectivity of the similar map introduced in [KL09] is essentially due to the existence of curved deformations. If one allows those back into the picture, the question becomes one of rectifying cA_∞ deformations to cdg equivalent ones. The defect in injectivity is fixed by (forcibly, since the notion from [KL09] does not apply to the curved case) changing the notion of equivalence of deformations, making more coarse.

5.2.1 Hochschild cohomology and Morita bimodules

Let A, B be dg algebras over k and X an A-B bimodule. Following [KL09], we will use the *arrow category* $\mathfrak{c}_X$ associated to the bimodule X, which is a dg category with two distinct objects P and Q and

$$\mathrm{Hom}_{\mathfrak{c}_X}(P,P) = A,\ \mathrm{Hom}_{\mathfrak{c}_X}(Q,Q) = B,\ \mathrm{Hom}_{\mathfrak{c}_X}(Q,P) = X \text{ and } \mathrm{Hom}_{\mathfrak{c}_X}(P,Q) = 0.$$

with composition defined in the obvious way. The category $\mathfrak{c}_X$ comes equipped with two fully faithful dg functors

$$j \colon A \to \mathfrak{c}_X \text{ and } i \colon B \to \mathfrak{c}_X$$

where A and B are seen as dg categories with one object, which are identified respectively with P and Q. In [Kel03], Keller introduced for any Morita bimodule X a bijection[2]

$$\varphi_X \colon\ \mathrm{HH}^\bullet(B) \to \mathrm{HH}^\bullet(A)$$

between the Hochschild cohomologies of B and A which is functorial with respect to the (derived) tensor product of bimodules. By construction of φ_X, the diagram

$$\begin{tikzcd}
 & \mathrm{HH}^\bullet(B) \arrow[dd, "\varphi_X"] \\
\mathrm{HH}^\bullet(\mathfrak{c}_X) \arrow[ur, "i^*"] \arrow[dr, "j^*"'] & \\
 & \mathrm{HH}^\bullet(A)
\end{tikzcd}$$

is a commutative diagram of isomorphisms, where j^*, i^* are the maps induced by the restrictions along the fully faithful functors $A \to \mathfrak{c}_X$ and $B \to \mathfrak{c}_X$.

5.2.2 Curved Morita deformations

Denote with $k[\varepsilon]$ the algebra of the dual numbers. Let A be a dg algebra over k. In the following, we will just say $k[\varepsilon]$-free to mean free as a graded $k[\varepsilon]$-module.

[2] In fact φ_X is shown to give an isomorphism in the homotopy category of B_∞ algebras between the Hochschild complexes, but we will only need the result at the homotopy level.

Definition 5.2.2. A curved Morita deformation of A is a $k[\varepsilon]$-free cdg $k[\varepsilon]$-algebra B_ε equipped with, setting $B := B_\varepsilon \otimes_{k[\varepsilon]} k$, a B-A Morita bimodule X. Two curved deformations (B_ε, X) and (C_ε, Y) are equivalent if there exists a cdg B_ε-C_ε bimodule Z_ε that is free as a graded B_ε-module and as a graded C_ε-module (in particular, it is $k[\varepsilon]$-free) and such that, setting $C = C_\varepsilon \otimes_{k[\varepsilon]} k$, the B-C bimodule $Z = Z_\varepsilon \otimes_{k[\varepsilon]} k$ is cofibrant as a bimodule and there exists an isomorphism $X \cong Z \otimes_C Y$ in the derived category of B-A bimodules.

We will denote with $\mathrm{cDef}_A(k[\varepsilon])$ the set of curved Morita deformations of A up to equivalence.

Lemma 5.2.3. *Equivalence of curved deformations is an equivalence relation, so the set* $\mathrm{cDef}_A(k[\varepsilon])$ *is well defined.*

Proof. Transitivity is easy, since if a deformation B_ε is equivalent via X_ε to C_ε and C_ε is equivalent to D_ε via Y_ε then one can check that $X_\varepsilon \otimes_{C_\varepsilon} Y_\varepsilon$ gives an equivalence between B_ε and D_ε. Symmetry is less immediate: let Z_ε be a morphism between deformations B_ε and C_ε. Then Z is a Morita bimodule, thus there exists a cofibrant C-B Morita bimodule W such that $Z \otimes_C W \cong B$ and $W \otimes_B Z \cong C$. Denoting with $\mathfrak{c}_W$ the arrow category of W, by definition of φ_W there is a commutative diagram of isomorphisms

$$\begin{tikzcd} & \mathrm{HH}^2(C) \arrow[dd, "\varphi_W"] \\ \mathrm{HH}^2(\mathfrak{c}_W) \arrow[ur, "i^*"] \arrow[dr, "j^*"'] & \\ & \mathrm{HH}^2(B). \end{tikzcd}$$

Since φ_W is an inverse to φ_Z and by Proposition 5.2.4 the map φ_Z carries the class that defines B_ε to the one defining C_ε, reasoning exactly as in the proof of Proposition 5.2.6 we can conclude that there exists a graded B_ε-free and C_ε-free C_ε-B_ε cdg bimodule $\hat{W}_\varepsilon$ such that its reduction $\hat{W}$ is cofibrant and isomorphic in the derived category of bimodules to W, which is a Morita equivalence. We have thus proved that equivalence of curved deformations is a symmetric relation. Reflexivity is proven in the same way: for any deformation B_ε we have the cdg B_ε-bimodule B_ε whose reduction B is manifestly a Morita bimodule; resolving this via the same procedure as before, we are done. □

Remark. It is clear that the set $\mathrm{cDef}_A(k[\varepsilon])$ is really just the set of connected components of a (higher) groupoid. A satisfactory construction of the whole groupoid would require a generalization to curved categories of the Morita theory of [Toë07] where the weak equivalences are, at least on $k[\varepsilon]$-free categories, the morphisms inducing weak equivalence on the (uncurved) reductions. In this generalization, the mapping spaces would be given by (an enhancement of) some version of the semiderived category of bimodules, following constructions in [Pos18]; indeed our condition of graded C_ε-freeness and cofibrancy of the reduction is implied by the natural cofibrancy condition for the semiderived category of [Pos18]. This is also related to the 1-derived category, since its subcategory given by the $k[\varepsilon]$-free modules coincides with the semiderived category. At the moment, this theory does not exist so we have to define our objects "by hand". In particular, we have no intrinsic definition of the left derived functor of the reduction $- \otimes_{k[\varepsilon]} k$ from (weakly) curved dg $k[\varepsilon]$-algebras to dg k-algebras; we solve this issue by restricting to $k[\varepsilon]$-free algebras to begin with.

5.2.3 Curved deformations and Hochschild cohomology

It is well-known (see [Low08; KL09]) that there is a correspondence between the set of Hochschild 2-cocycles and that of deformations of A as a cA_∞ algebra, i.e. $k[\varepsilon]$-free cA_∞ algebras A_ε equipped with an isomorphism of dg algebras $A_\varepsilon \otimes_{k[\varepsilon]} k \cong A$. Define a map

$$\nu\colon \operatorname{cDef}_A(k[\varepsilon]) \to \mathrm{HH}^2(A)$$

in the same way as in [KL09]: if (B_ε, X) is a curved deformation of A, then B_ε is a cA_∞ deformation of B, so defines a cocycle η and a class $[\eta]$ in $\mathrm{HH}^2(B)$. The class $\nu(B_\varepsilon) \in \mathrm{HH}^2(A)$ is given by definition by $\varphi_X([\eta])$.

Proposition 5.2.4. *The map ν is well defined.*

Proof. The proof of [KL09, Proposition 3.3] applies verbatim. □

Proposition 5.2.5. *The map ν is surjective.*

Proof. Let $[\eta] \in \mathrm{HH}^2(A)$ be an Hochschild class; then any cocycle η representing the class defines a cA_∞ deformation A_ε of A. Assume for now that there exists a $k[\varepsilon]$-free cdg algebra B_ε equipped with a cA_∞ B_ε-A_ε bimodule Z_ε such that the reduction Z is a Morita B-A bimodule (in particular, cannot have higher components). It is clear that (B_ε, Z) is an element of $\operatorname{cDef}_A(k[\varepsilon])$. By definition of φ_Z, the diagram

$$\begin{tikzcd} & \mathrm{HH}^2(B) \arrow[dd, "\varphi_Z"] \\ \mathrm{HH}^2(\mathfrak{c}_Z) \arrow[ur, "i^*"] \arrow[dr, "j^*"'] & \\ & \mathrm{HH}^2(A) \end{tikzcd}$$

commutes, and $\mathfrak{c}_{Z_\varepsilon}$ defines a cA_∞ deformation of $\mathfrak{c}_Z$ which defines an Hochschild cocycle $\mu \in \mathrm{HH}^2(\mathfrak{c}_Z)$. Moreover by construction, $j^*[\mu] = [\eta]$; therefore, $\nu(B_\varepsilon) = j^*[\mu] = [\eta]$. We are left to prove that such B_ε exists. Consider the cdg algebra $\mathcal{Y}(A_\varepsilon)$ given as the image via the curved Yoneda embedding (see [DL18] for a complete description of the curved Yoneda embedding and of the category of qA_∞ modules) of the cA_∞ algebra A_ε. This is isomorphic to the cdg algebra $\operatorname{Hom}_{A_\varepsilon}(A_\varepsilon, A_\varepsilon)$ where the hom is taken in the cdg category of qA_∞ A_ε-modules. Since A_ε is $k[\varepsilon]$-free, the same holds for $\mathcal{Y}(A_\varepsilon)$ since, as a graded $k[\varepsilon]$-module, it is a product of homs between $k[\varepsilon]$-free modules. There is a natural map of cA_∞ algebras $A_\varepsilon \xrightarrow{\mathcal{Y}} \mathcal{Y}(A_\varepsilon)$ given by the curved Yoneda embedding. Its higher components are killed by reduction, so setting $\mathcal{Y}(A_\varepsilon)_0 := \mathcal{Y}(A_\varepsilon) \otimes_{k[\varepsilon]} k$, we have a morphism of dg algebras $A \xrightarrow{\mathcal{Y}_0} \mathcal{Y}(A_\varepsilon)_0$. This coincides with the A_∞ Yoneda embedding which, by [DL18, Theorem 4.15] is a quasi-isomorphism and we are done. □

Proposition 5.2.6. *The map ν is injective.*

Proof. Suppose that (B_ε, X) and (C_ε, Y) are cdg deformations of A such that $\nu(B_\varepsilon) = \nu(C_\varepsilon)$. Let Z be a cofibrant Morita B-C bimodule such that there is an isomorphism $Z \otimes_C Y \cong X$ in the derived category – such a bimodule always exists by the standard Morita theory of dg algebras, see e.g. the proof of [KL09, Proposition 3.7]. Then by definition of φ_Z the diagram

$$\begin{tikzcd} & \mathrm{HH}^2(B) \arrow[dr, "\varphi_X"] & \\ \mathrm{HH}^2(\mathfrak{c}_Z) \arrow[ur, "i^*"] \arrow[dr, "j^*"'] & & \mathrm{HH}^2(A) \\ & \mathrm{HH}^2(C) \arrow[ur, "\varphi_Y"'] & \end{tikzcd}$$

is commutative. Denoting with η_B and η_C the Hochschild cocycles of B and C defining their cA_∞ deformations B_ε and C_ε, since they map to the same element in $\mathrm{HH}^2(A)$ and all arrows are isomorphisms, there must be an element $[\gamma] \in \mathrm{HH}^2(\mathfrak{c}_Z)$ such that

$$i^*([\gamma]) = [\eta_{\mathcal{B}}] \text{ and } j^*([\gamma]) = [\eta_{\mathcal{C}}].$$

By [KL09, Lemma 3.8], it is actually possible to find a 2-cocycle γ mapping to $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ before passing to cohomology. Now γ defines a cA_∞ deformation of $\mathfrak{c}_Z$ which is immediately seen to be itself an arrow category for some B_ε-C_ε cA_∞ bimodule Z_ε. By construction of Z_ε it is $k[\varepsilon]$-free and its reduction modulo ε is the B-C Morita bimodule Z.

We are almost done, except that we need to rectify Z_ε to a cdg – and not cA_∞ – bimodule. This is an application of Koszul duality, for which we employ the notations and constructions of [Pos11, Sections 6 and 8][3]. Using the fact that the (co)bar construction is appropriately monoidal, a cA_∞ B_ε-C_ε bimodule corresponds to a cA_∞ module over the cdg algebra $E_\varepsilon = {B_\varepsilon}^{\mathrm{op}} \otimes_{k[\varepsilon]} C_\varepsilon$. This by definition is a cdg comodule $\mathrm{Bar}_v(E_\varepsilon, Z_\varepsilon)$ over the cdg coalgebra $\mathrm{Bar}_v(E_\varepsilon)$ (for a similar construction, see the curved bar construction of [Pos18]). Consider then the cdg E_ε-module $\hat{Z}_\varepsilon = E_\varepsilon \otimes^\tau \mathrm{Bar}_v(E_\varepsilon, Z_\varepsilon)$. This is graded E_ε-free – its underlying graded module is $E_\varepsilon \otimes_{k[\varepsilon]} \mathrm{Bar}_v(E_\varepsilon, Z_\varepsilon)$ – so it is both graded C_ε-free and graded B_ε-free. Moreover, denoting $E = E_\varepsilon \otimes_{k[\varepsilon]} k \cong B^{\mathrm{op}} \otimes_k C$, the reduction $\hat{Z}$ of $\hat{Z}_\varepsilon$ is the dg E-module $E \otimes^\tau \mathrm{Bar}_v(E, Z)$, where now the tensors are over the base field k. By the proof of [Pos11, Theorem 6.3], this coincides with the (reduced) bar resolution of the dg E-module M, which is therefore both cofibrant and, being isomorphic to Z in the derived category of bimodules, a Morita equivalence. □

[3]In principle the constructions there are only given for a base field, but since in our case everything is (graded) $k[\varepsilon]$-free we can repeat verbatim his constructions over this base ring. In particular, all the tensor products in the various bar constructions are intended over $k[\varepsilon]$.

5.3 A commutative square of deformations

Finally, we can show that the map $\mu_{D(A)}$ is compatible with the bijection

$$\nu\colon \operatorname{cDef}_A(k[\varepsilon]) \to \mathrm{HH}^2(A)$$

introduced in Section 5.2, yielding the square (1.4) promised in the introduction (Theorem 5.3.2).

5.3.1 Curved Morita deformations and Hochschild cohomology

Recall that if A, B are dg algebras, an A-B-bimodule X is said to be a Morita equivalence if the induced adjoint pair

$$D(A) \underset{\mathbb{R}\operatorname{Hom}_B(X,-)}{\overset{-\otimes^{\mathrm{L}}_A X}{\rightleftarrows}} D(B)$$

is an equivalence. For notational simplicity, denote with F_X the equivalence $-\otimes^{\mathrm{L}}_A X$ and with F_X^{-1} its inverse $\mathbb{R}\operatorname{Hom}_B(X,-)$.

5.3.2 From algebras to categories

We start with the following:

Proposition 5.3.1. *For any curved Morita deformation (B_ε, X) of A, the 1-derived category $D^\varepsilon(B_\varepsilon)$ is a categorical deformation of $D(A)$. Equivalent Morita deformations yield equivalent categorical deformations, so this assignment gives a well-defined map*

$$D^\varepsilon(-)\colon \operatorname{cDef}_A(k[\varepsilon]) \to \operatorname{CatDef}_{D(A)}(k[\varepsilon])$$

Proof. By Theorem 5.1.1, the category $D^\varepsilon(B_\varepsilon)$ is a categorical deformation of $D(B)$, i.e. there exists a recollement

$$\begin{tikzcd} D(B) \arrow[r, "i" description] & D^\varepsilon(B_\varepsilon) \arrow[l, bend right, "Q"'] \arrow[l, bend left, "K"] \arrow[r, "E" description] & D(B) \arrow[l, bend right, "G"'] \arrow[l, bend left] \end{tikzcd}$$

together with two triangles

$$E \to K \to C \to E[1] \text{ and } C \to Q \to E \to C[1]. \tag{5.8}$$

We can thus define

$$i_A = iF_X,\ G_A = GF_X,\ E_A = F_X^{-1}E,\ K_A = F_X^{-1}L,\ Q_A = F_X^{-1}Q$$

to obtain a recollement

$$\begin{tikzcd} D(A) \arrow[r, "i_A" description] & D^\varepsilon(B_\varepsilon) \arrow[l, bend right, "Q_A"'] \arrow[l, bend left, "K_A"] \arrow[r, "E_A" description] & D(A); \arrow[l, bend right, "G_A"'] \arrow[l, bend left] \end{tikzcd}$$

similarly one can compose the triangles (5.8) with F_X^{-1} to obtain the relevant Yoneda extension, and thus show that $D^\varepsilon(B_\varepsilon)$ is indeed a deformation of $D(A)$.

To show that equivalent curved deformations yield equivalent categorical deformations, assume that Z_ε is an equivalence between two Morita deformations B_ε and C_ε. Then by Proposition 3.4.3 there is an induced equivalence $D^\varepsilon(B_\varepsilon) \cong D^\varepsilon(C_\varepsilon)$ which, by Lemma 3.4.5 is compatible with the semiorthogonal decompositions; it is straightforward to check that this is also an equivalence of deformations – this essentially follows from the fact that the equivalences are $k[\varepsilon]$-linear and the Yoneda extensions are defined in terms of the action of t. $\square$

We can now tie everything together:

Theorem 5.3.2. *Let A be a dg algebra. There is a commutative square of bijections*

$$\begin{tikzcd} \mathrm{cDef}_A(k[\varepsilon]) \arrow[r, "\nu"] \arrow[d, "D^\varepsilon(-)"'] & \mathrm{HH}^2(A) \arrow[d, "\chi_A"] \\ \mathrm{CatDef}_{D(A)}(k[\varepsilon]) \arrow[r, "\mu_{D(A)}"] & \mathrm{HH}^2(D(A)) \end{tikzcd}$$

Proof. We already know the arrows χ_A, $\mu_{D(A)}$ and ν are bijections, so only the commutativity of the diagram remains to be shown. Let (B_ε, X) be a curved Morita deformation of A; then $D^\varepsilon(B_\varepsilon)$ is a categorical deformation of $D(A)$, and we want to compute its associated class in $\mathrm{HH}^2(D(A))$. The equivalence F_X defines a bijection $\mathrm{HH}^2(D(B)) \overset{\varphi_{F_X}}{\to} \mathrm{HH}^2(D(A))$: given a class

$$\mathrm{id}_{D(B)}[-1] \overset{\eta_B}{\to} \mathrm{id}_{D(B)}[1]$$

in $\mathrm{HH}^2(D(B))$, we define $\varphi_{F_X}(\eta_B)$ as the class

$$\mathrm{id}_{D(A)}[-1] \cong F_X^{-1}\,\mathrm{id}_B\,F_X[-1] \overset{F_X^{-1}\eta_B F_X}{\to} F_X^{-1}\,\mathrm{id}_B\,F_X[-1] \cong \mathrm{id}_{D(A)}[-1]$$

in $\mathrm{HH}^2(D(A))$; this is compatible with the morphism χ_A, in the sense that the diagram

$$\begin{tikzcd} \mathrm{HH}^2(B) \arrow[r, "\varphi_X"] \arrow[d, "\chi_B"'] & \mathrm{HH}^2(A) \arrow[d, "\chi_A"] \\ \mathrm{HH}^2(D(B)) \arrow[r, "\varphi_{F_X}"] & \mathrm{HH}^2(D(B)) \end{tikzcd}$$

commutes. Recall that $\mu_B \in \mathrm{HH}^2(B)$ is the class corresponding to the cdg deformation B_ε of B. By construction, one has that

$$\mu_{D(A)}(D^\varepsilon(B_\varepsilon)) = \varphi_{F_X} \circ \mu_{D(B)}(D^\varepsilon(B_\varepsilon)).$$

Hence, by Theorem 5.1.1, we get

$$\mu_{D(A)}(D^\varepsilon(B_\varepsilon)) = \varphi_{F_X}(\chi_B(\mu_B)) = \chi_A(\varphi_X(\mu_B)) = \chi_A(\nu(B_\varepsilon))$$

and we are done. $\square$

5.4 Deformations and smoothness

As an application we show that, unlike in the classical case, smoothness is preserved under categorical deformations. Recall from [KL12] that a dg category $\mathcal{C}$ is said to be *smooth* if the diagonal bimodule $\mathcal{C} \in D(\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C})$ is compact[4].

Proposition 5.4.1. *Let $\mathcal{T}_\varepsilon$ be a categorical deformation of a triangulated category $\mathcal{T}$. Then $\mathcal{T}_\varepsilon$ is smooth if and only if $\mathcal{T}$ is smooth; the same statement holds for properness*[5].

Proof. One implication follows directly from [KL12, Proposition 4.9], since $\mathcal{T}$ is the only semiorthogonal factor of $\mathcal{T}_\varepsilon$. For the other implication, we know again by [KL12, Proposition 4.9] that, being $\mathcal{T}$ smooth, the gluing $\mathcal{T}_\varepsilon$ is smooth if and only if the gluing bimodule $KG[1] \in D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$ is perfect. Since $\mathcal{T}$ is smooth, the diagonal bimodule $\mathrm{id}_{\mathcal{T}} \in D(\mathcal{T}^{\mathrm{op}} \otimes \mathcal{T})$ is perfect. From the existence of the triangle

$$\mathrm{id}_{\mathcal{T}}[-1] \overset{\mu_{\mathcal{T}}(\mathcal{T}_\varepsilon)}{\to} \mathrm{id}_{\mathcal{T}}[1] \to KG[1] \to \mathrm{id}_{\mathcal{T}}$$

of Corollary 4.1.3 we deduce that $KG[1]$ is also perfect. The same argument holds for properness. □

In fact, the proof says something more: the functor KG is obtained as a finite extension of copies of $\mathrm{id}_{\mathcal{T}}$, so it is "as perfect as" $\mathrm{id}_{\mathcal{T}}$. This, in turn, conveys the idea that $\mathcal{T}_\varepsilon$ is "as singular as" $\mathcal{T}$. Consider now the case of a deformation A_ε of a dg algebra A, and assume that A_n is a honest dg algebra (i.e. has no curvature).

One sees that if A is a smooth dg algebra, then its (dg enhanced) derived category $D(A)$ is smooth. On the other hand, even when A is smooth, the deformation A_ε is never smooth and thus neither is $D(A_\varepsilon)$; this is due to the fact that the base ring R_n is not reduced and thus is not itself smooth. Recall that a *categorical resolution* of a pretriangulated dg category $\mathcal{A}$ is a smooth pretriangulated dg category $\mathcal{C}$ equipped with a dg functor $\mathcal{A} \to \mathcal{C}$ which induces a fully faithful functor between the homotopy categories. We can therefore prove the following fact:

Proposition 5.4.2. *If A is a homologically smooth dg algebra and the deformation A_ε is a dg algebra, then $D^\varepsilon(A_\varepsilon)$ is a categorical resolution of $D(A_\varepsilon)$.*

Proof. Fist of all, observe that since A is homologically smooth the same holds for $D(A)$. We know from Corollary 3.1.10 that there is an embedding

$$D(A_\varepsilon) \hookrightarrow D^\varepsilon(A_\varepsilon);$$

since $D^\varepsilon(A_\varepsilon)$ is a categorical deformation of $D(A)$, by Proposition 5.4.1 it is homologically smooth and hence a categorical resolution of $D(A_\varepsilon)$. □

[4] One should be careful about size issues, which we ignore here; for an in depth treatment, see [LO10, Appendix A].

[5] A dg category $\mathcal{A}$ is said to be proper if for all $A, B \in \mathcal{A}$, the k-module $\oplus_i H^i \operatorname{Hom}_{\mathcal{A}}(A, B)$ is finite-dimensional.

It is reasonable to see $D^\varepsilon(A_\varepsilon)$ as a *blowup* of $D(A_\varepsilon)$. Indeed, the procedure of moving from $D(A_\varepsilon)$ to $D^\varepsilon(A_\varepsilon)$ only resolves the singularity coming from the presence of the nilpotent deformation parameter; any singularity "away from 0" will still be present in $D^\varepsilon(A_\varepsilon)$. This procedure offers some insight into the relation between the classical notion of deformation for triangulated categories ([Lur11; GLSVdB24; BKP18]) and our categorical deformations. The point is that, given a classical deformation, its blowup (in an appropriate sense) is a categorical deformation. One can use this setup to investigate the natural question of whether – in specific cases – classical deformations span the whole Hochschild complex and, in cases where they don't, which part of the Hochschild complex they do span. This question can be reformulated as asking which classes of categorical deformations can be obtained as blowups of classical deformations. This is useful in practice, since it allows one access to properties and invariants of the categorical deformation. In future work we will further explore this perspective, in particular in relation to the results from [GLSVdB24].

5.5 Extending A_∞-functors

In this section we record some technical results about extensions of A_∞-functors that will be needed in the later section (Proposition 5.5.1); we also observe a relation between this construction and Hochschild cohomology.

5.5.1 Extension of functors

If $\mathcal{A}$ and $\mathcal{B}$ are pretriangulated dg categories, denote with $\mathrm{Fun}_{A_\infty}(\mathcal{A},\mathcal{B})$ the dg category of A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$ [COS19]; similarly, if $\mathcal{A}$ and $\mathcal{B}$ admit small coproducts, denote with $\mathrm{Fun}^c_{A_\infty}(\mathcal{A},\mathcal{B})$ the dg category of cocontinuous A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$ - that is, A_∞-functors for which the underlying $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ preserves small coproducts. It is a well-known fact (see [LRG22, Proposition 3.27]) that the restriction dg functor

$$\mathrm{Fun}^c_{A_\infty}(D(A),\mathcal{B}) \to \mathrm{Fun}_{A_\infty}(A,\mathcal{B})$$

along the Yoneda embedding $A \to D(A)$ is a quasi-equivalence. The enhancement $\mathrm{Tw}(A)$ allows for an easy description of a quasi-inverse to the restriction, given by the natural extension of an A_∞-functor to the category of twisted objects. This is well-known and often used implicitly in the literature (see e.g. [Hai24, Section 2]); similar computations also appear in [AL17; AL21]. For future use, we describe explicitly the formulas appearing in the extension:

Proposition 5.5.1. *Any A_∞-functor $F\colon A \to \mathcal{B}$ can be canonically extended to a cocontinuous A_∞-functor*

$$\chi_A(F)\colon D(A) \to \mathcal{B}.$$

This assignment provides a dg functor

$$\chi_A\colon\ \mathrm{Fun}_{A_\infty}(A,\mathcal{B}) \to \mathrm{Fun}_{A_\infty}(D(A),\mathcal{B})$$

which preserves triangles in the homotopy category.

Construction of the extension. Preliminarily, observe that since $\mathcal{B}$ is pretriangulated and closed under coproducts, there is a fully faithful totalization dg functor

$$\mathrm{Tw}(\mathcal{B}) \to \mathcal{B}.$$

Hence, it will be enough to describe $\chi_A(F)$ as a functor $\mathrm{Tw}(A) \to \mathrm{Tw}(\mathcal{B})$. For simplicity we omit the signs from the formulas; the interested reader can consult [Low08] for the precise sign conventions. By definition, f is given by an object $FA \in \mathcal{B}$ together with an A_∞-algebra morphism (see Section 5.1.2)

$$A \to \mathrm{Hom}_{\mathcal{B}}(FA, FA)$$

given by components $F_i \colon A^{\otimes i} \to \mathrm{Hom}_{\mathcal{B}}(FA, FA)$. Given $M = (\oplus A_i[n_i], \delta_M)$ in $\mathrm{Tw}(A)$ set

$$F(\delta_M) = \sum_i F_i(\delta_M^{\otimes i})$$

and define $\chi_A(F)(M) = (\oplus_i fA[n_i], F(\delta))$. Given

$$M = (\oplus_{i\in I} A[n_i], \delta_M),\ N = (\oplus_{i\in J} A[n_j], \delta_N) \text{ and } g \in \mathrm{Hom}_{\mathrm{Tw}(A)}(M, N),$$

define then

$$\chi_A(F)_1(g) = \sum_{i\geq 1} F_i(\delta^k \otimes g \otimes \delta^{\otimes l});$$

in general, given $M_0, \dots M_n \in \mathrm{Tw}(A)$ and $g^i \in \mathrm{Hom}_{\mathrm{Tw}(A)}(M_{i-1}, M_i)$ define the component $\chi_A(F)_n(g^1, \dots g^n)$ as

$$\sum_{i\geq n} \sum_{k_0+\dots k_n = i-n} f_i(\delta^{\otimes k_0} \otimes g^1 \otimes \delta^{\otimes k_1} \otimes g^2 \otimes \dots \otimes \delta^{\otimes k_{n-1}} \otimes g^n \otimes \delta^{\otimes k_n}).$$

One verifies that these formulas define an A_∞-functor. To enhance $\chi_A(-)$ to a dg functor, one must define an action of natural transformation, that is: given two functors

$$f, g \colon A \to \mathcal{B}$$

and an A_∞ natural transformation $\eta \colon f \to g$ given by components

$$\eta_i \colon A^{\otimes i} \to \mathrm{Hom}_{\mathcal{B}}(fA, hA)$$

we want to construct an A_∞-natural transformation $\chi_A(f) \to \chi_A(h)$. We define the component $\chi_A(\eta)_n(g^1, \dots g^n)$ as

$$\sum_{i\geq n} \sum_{k_0+\dots k_n = i-n} \eta_i(\delta^{\otimes k_0} \otimes g^1 \otimes \delta^{\otimes k_1} \otimes g^2 \otimes \dots \otimes \delta^{\otimes k_{n-1}} \otimes g^n \otimes \delta^{\otimes k_n}).$$

One verifies that χ_A commutes (strictly) with the composition of natural transformations and with the differentials, hence it defines a dg (and not A_∞!) functor

$$\chi_A \colon \mathrm{Fun}_{A_\infty}(A, \mathcal{B}) \to \mathrm{Fun}_{A_\infty}(D(A), \mathcal{B}).$$

$\square$

5.5.2 The characteristic morphism

As already noted, the formulas from the previous section are essentially the same appearing in [Low08]; let us make this fact precise. Let $C^\bullet(A)$ be the Hochschild cochain complex of A. It is well-known that there is an isomorphism

$$C^\bullet(A) \cong \mathrm{Hom}_{\mathrm{Fun}_{A_\infty}(A,A)}(\mathrm{id}_A, \mathrm{id}_A),$$

where A is viewed as a dg category with one object, and id_A is the identity dg functor $A \overset{\mathrm{id}_A}{\to} A$. At the same time, there is a fully faithful embedding

$$\mathrm{Fun}_{A_\infty}(A, A) \hookrightarrow \mathrm{Fun}_{A_\infty}(A, D(A))$$

induced by the embedding $A \overset{\mathrm{inc}}{\hookrightarrow} D(A)$. The extension via χ_A of the inclusion $A \to D(A)$ is the identity functor of $D(A)$, hence the image of id_A via the composition

$$\mathrm{Fun}_{A_\infty}(A, A) \hookrightarrow \mathrm{Fun}_{A_\infty}(A, D(A)) \overset{\chi_A}{\to} \mathrm{Fun}_{A_\infty}(D(A), D(A))$$

is $\mathrm{id}_{D(A)}$. Therefore, the functor χ_A defines a map

$$C^\bullet(A) \cong \mathrm{Hom}_{\mathrm{Fun}_{A_\infty}(A,A)}(\mathrm{id}_A, \mathrm{id}_A) \to \mathrm{Hom}_{\mathrm{Fun}_{A_\infty}(D(A),D(A))}(\mathrm{id}_{D(A)}, \mathrm{id}_{D(A)}) \cong C^\bullet(D(A)).$$

This coincides with the map χ_A defined in [Low08].

Chapter 6

Future research prospects

In this short chapter, we discuss some potential implications and improvements of the results presented in the rest of the thesis.

6.1 Coherent complexes and higher actions

As already discussed, our definition of categorical deformation only requires working with the homotopy categories of the various functor categories, without considering their enhancements. This allowed us to give fairly direct constructions, but also has its downsides. Our theory is well-equipped to decide whether two deformations are equivalent, but less so to describe the group of autoequivalences of a deformation. A notion of morphism of deformation more apt to this problem would entail having a functor which induces in an appropriate sense a morphism of Yoneda extensions (see Proposition 2.1.3). The issue is that, in our setting, it is unreasonable for a functor to commute with the boundary object C, since that is often only defined up to a noncanonical isomorphism. For this construction to work, one would need a fully enhanced notion of categorical deformation.

Indeed, one would want to define a Yoneda extension of functors as a sequence

$$0 \to E \xrightarrow{\delta_1} K \xrightarrow{\alpha} Q \xrightarrow{\delta_2} E \to 0$$

with $\alpha\delta_1 = \delta_2\alpha = 0$ with the property that α induces an isomorphism $\bar{\alpha}$ between the cone C of δ_1 and the cocone D of δ_2, as in

$$\begin{tikzcd} E \arrow[r, "\delta_1"] & K \arrow[r] \arrow[d, "\alpha"'] & C \arrow[dll, dashed, bend left, "\bar{\alpha}"] \\ D \arrow[r] & Q \arrow[r, "\delta_2"] & E. \end{tikzcd}$$

Even though one can show that $\alpha\delta_1 = \delta_2\alpha = 0$ the issue is that, if we remain in the homotopy category, the Toda bracket $\langle \delta_1, \alpha, \delta_2 \rangle$ is an obstruction to the existence of any map $\bar{\alpha}$ as in the diagram. This problem can be solved by committing to the higher categorical world; indeed, sequences with vanishing Toda Brackets are known [Ari22] to correspond to *coherent complexes*. To be more precise, one can consider the data of a recollement, together with the following morphisms in the (enhanced) functor category:

- A sequence of morphisms

$$0 \to E \stackrel{\delta_1}{\to} K \stackrel{\alpha}{\to} Q \stackrel{\delta_2}{\to} E \to 0; \tag{6.1}$$

- A nullhomotopy t for the composition $\alpha\delta_1$ and a nullhomotopy s for $\delta_2\alpha$;
- A homotopy H between $\delta_2 t$ and $s\delta_1$.

This data defines a canonical map $\bar{\alpha}$ between the cone of δ_1 and the cocone of δ_2; a categorical deformation would then be given by the above data, under the condition for $\bar{\alpha}$ to be an equivalence; note that this is a way to express the "exactness" of the sequence (6.1). Any such construction yields, by passing to the homotopy category, a Yoneda extension in the sense of Section 2.1.4. Highlighting the role of the natural transformations δ_1 and δ_2 also hints at the sense in which the category $\mathcal{T}_\varepsilon$ ought to be considered $k[\varepsilon]$-linear. Indeed, under the adjunctions between i and K and Q, the two natural transformations

$$E \stackrel{\delta_2}{\to} K \text{ and } Q \stackrel{\delta_2}{\to} E$$

correspond to natural transformations

$$iE \to \mathrm{id}_{\mathcal{T}_\varepsilon} \to iE.$$

This should be though as the categorification of the two natural transformations

$$tM \hookrightarrow M \stackrel{t}{\to} tM$$

for a $k[\varepsilon]$-module M; hence, $k[\varepsilon]$ "acts" on the category $\mathcal{T}_\varepsilon$ via the action of the endofunctor iE, with the various natural transformations functioning as compatibility conditions.

In order to work effectively with these objects, however, our framework has to be recast in the language of ∞-category theory; a full treatment of these, more delicate, higher aspects is thus postponed to future work.

We conclude by observing that further, leaving the model of dg categories would allow to give a fairly straightforward description of *absolute*, or non-linear deformations in terms of topological Hochschild cohomology, which should in turn be compared with [KL11, Section 5][KLBa; KLBb].

6.2 Formal deformation theory

One way to interpret the results of this thesis is as describing the tangent space – via first order deformations – to a certain, not fully understood, moduli space of triangulated categories. At the local level, it is a well known maxim that "deformation problems correspond to dg Lie algebras". This statement can be formalized in the setting of derived deformation theory, and has been shown independently by Pridham in [Pri10] and Lurie in [Lur11]. Here, a deformation problem is encoded via an (enhanced) functor $\mathbf{dgart}_k \to \mathcal{S}$ from the category of artinian dg algebras to the category of spaces, which satisfies some formal properties. The precise statement is then given as an equivalence of (homotopy) categories between the category of deformation functors and the category of dg Lie algebras. In this framework the curvature problem corresponds to the fact that, for a fixed (enhanced) triangulated category $\mathcal{C}$, the functor

$$\mathrm{CatDef}_{\mathcal{C}}(-)\colon \mathbf{dgart}_k \to \mathcal{S}$$

which assigns to an algebra R the set of classical R-deformations of $\mathcal{C}$ does *not* satisfy the formal properties required of a deformation functor. On the other hand, the theory yields a "best approximation" deformation functor

$$\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(-)\colon \mathbf{dgart}_k \to \mathcal{S}$$

which comes equipped with a universal natural transformation $\mathrm{CatDef}_{\mathcal{C}}(-) \xrightarrow{\eta} \mathrm{CatDef}^{\wedge}_{\mathcal{C}}(-)$; It was shown in [Lur11] that the functor $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(-)$ does correspond, under the equivalence between deformation problems and dglas, to the (shifted) Hochschild complex. One can easily see that $\pi_0 \,\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(k[\varepsilon]) \cong \mathrm{HH}^2(\mathcal{C})$, hence the transformation η recovers the usual map between (Morita) classical deformations and HH^2; the failure of this map to be a bijection is, as we have discussed, an instance of the curvature problem. The definition of the functor $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(-)$ is however somewhat unsatisfying, being constructed essentially via the Hochschild complex; no interpretation of the space $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(R)$ as a space of R-deformation of $\mathcal{C}$ is known. We then conjecture that the functor $\mathrm{CatDef}^{\wedge}_{\mathcal{C}}(-)$ parametrizes appropriately defined *categorical* deformations of $\mathcal{C}$. The results of Chapter 4, indeed, correspond to the first-order case of this statement. A full proof of this fact will however require extensive preliminary developments; first, as already discussed in Section 6.1, the theory would have to be recast in a more flexible homotopical framework. More crucially, a general notion of categorical deformation over general local artinian (dg) algebra must be developed. This is already a significant question, as the first-order case does not have an obvious generalization. Definitions for certain bases can be reasonably obtained – e.g., Theorem 3.5.7 suggests an inductive definition for categorical deformations over R_n – but the question of finding a definition that works for arbitrary bases is one that will be developed in future work.

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