Given a sequence $u_n \in \mathscr{S}'(\mathbb{R})$, we say that $u_n$ converges to $u \in \mathscr{S}'(\mathbb{R})$ if for any $\phi \in \mathscr{S}(\mathbb{R})$

$$u_n(\phi) \to u(\phi)$$

in $\mathbb{C}$. The goal of this exercise sheet is to compute the Fourier Transform of the step function $H$. In the following, given a function $u$, we will not distinguish between it and the associated distribution $T_u$;

**Exercise 1**

Define the distribution $\mathsf{pv}(\frac{1}{x})$ as

$$\mathsf{pv}(\frac{1}{x})(\phi) = \lim_{\epsilon \to 0^+} \int_{|x|>\epsilon} \frac{\phi(x)}{x} dx.$$

- Show that the formula is well defined, by proving that

$$\mathsf{pv}(\frac{1}{x})(\phi) = \int_{|x|>1} \frac{\phi(x)}{x} dx + \int_{|x<1} \frac{\phi(x) - \phi(0)}{x} dx;$$

- Show that $\mathsf{pv}(\frac{1}{x})$ lies in $\in \mathscr{S}'(\mathbb{R})$ (i.e. it is continuous), by proving the inequality

$$|\mathsf{pv}(\frac{1}{x})(\phi)| \leq 2||\phi'||_{L^\infty} + ||x\phi||_{L^\infty};$$

- Show that

$$\partial \log |x| = \mathsf{pv}(\frac{1}{x})$$

as a tempered distribution. You do not need to prove that $T_{\log |x|}$ is a tempered distribution. You want to integrate by parts, but be careful about the boundaries.

**Exercise 2**

- Show that

$$\frac{1}{2} \log(x^2 + \epsilon^2) \overset{\epsilon \to 0}{\to} \log |x| \text{ in } \mathscr{S}'(\mathbb{R});$$

- Use the point above to show that

$$\frac{x}{x^2 + \epsilon^2} \overset{\epsilon \to 0}{\to} \mathsf{pv}(\frac{1}{x}) \text{ in } \mathscr{S}'(\mathbb{R}).$$

**Exercise 3**

Recall the Heaviside step function defined as

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

- Show that
$$e^{-\epsilon x} H \overset{\epsilon \to 0}{\to} H \text{ in } \mathscr{S}'(\mathbb{R});$$
- Show that
$$\mathcal{F}(e^{-\epsilon x} H) = \frac{\epsilon}{\omega^2 + \epsilon^2} - i \frac{\omega}{\omega^2 + \epsilon^2}.$$

**Exercise 4** (Hard)

Show that

$$\frac{\epsilon}{x^2 + \epsilon^2} \to \pi \delta \text{ in } \mathscr{S}'(\mathbb{R});$$

Hint: fix an arbitrary $\sigma > 0$ (different than $\varepsilon$!) and split

$$\int \frac{\epsilon \phi(x)}{x^2 + \epsilon^2} dx = \int_{|x| < \sigma} \frac{\epsilon \phi(x)}{x^2 + \epsilon^2} dx + \int_{|x| > \sigma} \frac{\epsilon \phi(x)}{x^2 + \epsilon^2} dx$$

then send $\epsilon \to 0$.

**Exercise 5**

Conclude that

$$\mathcal{F} H = \pi \delta - i \mathsf{pv}(\frac{1}{\omega}).$$