Exercise 1

Determine whether these functions lie in the Schwartz space $\mathscr{S}(\mathbb{R})$:

$$|x|^{-12}, e^{-|x|}, e^{-|x|^2}.$$

Exercise 2

Recall that in the last session we defined a metric d in the Schwartz space $\mathscr{S}(\mathbb{R})$ with the property that a sequence ϕ_n converged to $\phi \in \mathscr{S}(\mathbb{R})$ precisely when

$$||x^a \partial^b (\phi_n - \phi)||_{L^\infty} \to 0$$

for all a, b. Show that the following operators $\mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ are continuous:

- The derivative ∂;
- The translation $\tau_h(\phi)(x) = \phi(x-h)$;
- The multiplication by a polynomial $p(x)$.
- The Fourier Transform $\mathcal{F}$.

Exercise 3

A *tempered distribution* is a continuous linear functional $u \colon \mathscr{S}(\mathbb{R}) \to \mathbb{C}$. The space of tempered distributions is denoted with $\mathscr{S}'(\mathbb{R})$.

It is a useful fact that a linear functional $u \colon \mathscr{S}(\mathbb{R}) \to \mathbb{C}$ is continuous if and only if there exist C, a, b such that

$$|u(\phi)| \leq p_N(\phi)$$

for all $\phi \in \mathscr{S}(\mathbb{R})$. Optional: prove this fact. If $u \colon \mathbb{R} \to \mathbb{C}$ is a function, define the operator

$$T_u \colon \mathscr{S}(\mathbb{R}) \to \mathbb{C}$$

as $T_u\phi = \int u\phi dx$. Show that the following operators are tempered distributions:

- T_u for u in the space $L^\infty(\mathbb{R})$ of bounded functions;
- T_u for u in $L^p(\mathbb{R})$ for $p \in [1, \infty)$. For this, you can use that for any Schwartz function ϕ and any $q \in [1, \infty]$ there exists an N such that $||\phi||_{L^q} \leq p_N(\phi)$.
- (Optional) T_u for u of the form $p(x)v(x)$ for a polynomial p and v in some L^p.
- The Dirac delta $\delta_0(\phi) = \phi(0)$.

- If u is a tempered distribution, its *distributional derivative*

$$\partial u(\phi) = -u(\partial\phi)$$

  and its translation

$$\tau_h u(\phi) = u(\tau_{-h})$$

- If u is a tempered distribution and $p(x)$ a polynomial, the multiplication

$$pu(\phi) = u(p(x)\phi).$$

Exercise 4

Let

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Show that T_H is a tempered distribution. Show that $\partial T_H = \delta_0$.

Exercise 5

Recall that if a function $\phi \in \mathscr{S}(\mathbb{R})$ then its Fourier Transform $\mathcal{F}\phi$ lies in $\mathscr{S}(\mathbb{R})$ as well. This means that if $u \in \mathscr{S}'(\mathbb{R})$ is a tempered distribution, we can define its Fourier transform

$$\mathcal{F}u(\phi) = u(\mathcal{F}\phi).$$

and similarly the conjugate Fourier transform

$$\overline{\mathcal{F}}u(\phi) = \overline{u(\mathcal{F}\phi)}.$$

Show that:

- $\overline{\mathcal{F}}\mathcal{F} = \mathcal{F}\overline{\mathcal{F}} = 2\pi\,\mathrm{Id}$;
- $\mathcal{F}(xu) = i\partial\mathcal{F}(u)$ and $\mathcal{F}(\partial u) = i\omega\mathcal{F}(u)$;
- $\mathcal{F}(\tau_h u) = e^{-ih\omega}\mathcal{F}(u)$ and $\mathcal{F}(e^{ihx}u) = \tau_h\mathcal{F}u$;

Compute the Fourier transforms of the following distributions:

- T_u for $u \in L^1(\mathbb{R})$;
- The Dirac delta δ_0;
- T_p for a polynomial $p(x)$ (Hint: what is $\mathcal{F}T_1$? What is $\mathcal{F}T_x$?);
- $T_{e^{ix}}$;
- $T_{\cos x}$.