Ex 1

Consider the function

$$f(x) = \begin{cases} \cos x & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0 & |x| \geq \frac{\pi}{2}. \end{cases}$$

You can assume that

$$\hat{f}(\omega) = \frac{2\cos(\frac{\pi}{2}\omega)}{\omega^2 - 1}$$

(optional: prove it!). Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos(\frac{\pi}{2}y)}{1-y^2} dy.$$

Ex 2

Show that if $f \in L^1(\mathbb{R})$, then

$$\lim_{\omega \to \infty} \hat{f}(\omega) = 0.$$

For this exercise you can assume the fact that $\mathscr{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ in the L^1 norm.

Ex 3

Show that the only possible eigenvalues of the fourier transform are

$$\{\sqrt{2\pi}, -\sqrt{2\pi}, \sqrt{2\pi}i, -\sqrt{2\pi}i\}.$$

Optional: show that these are indeed eigenvalues by finding the related eigenfunctions.

Ex 4

Let $f, g \in \mathscr{S}(\mathbb{R})$. Define for any $k \geq 0$

$$p_k(\phi) = \max_{|a|,|b|<k} ||x^a \partial^b \phi||_{L^\infty}$$

and

$$d(f,g) = \sum_{k \geq 0} 2^{-k} \frac{p_k(f-g)}{1 + p_k(f-g)}.$$

Show that

- d defines a metric on $\mathscr{S}(\mathbb{R})$.
- A sequence $\{f_n\}$ converges to f according to the metric d if and only if

$$p_j(f_n - f) \to 0 \qquad j \geq 0.$$