Exercise sessions 5 and 6

Algebraic Topology 2025-2026

24-25 March, 2026

Exercise* 1

Show that for any group G, there exists a path connected topological space X_G with the property that $\pi_1(X_G) = G$. Is this space unique?

Exercise 2

Compute the fundamental group of a torus using Van Kampen's theorem (that is, not using that $T^2 = S^1 \times S^1$). Compute the fundamental group of the Klein bottle.

Exercise 3

Compute the fundamental group of the real projective plane $\mathbb{RP}^2$.

Exercise 4

Prove or find a counterexample to the following statement: let X be a topological space which is the union of two contractible subspaces $A, B \subseteq X$ with contractible intersection. Then X is contractible.

Exercise* 5

Let G be a group. Recall that the abelianzation G^{ab} is defined as the quotient of G by $[G, G]$ generated by the commutators $aba^{-1}b^{b-1}$. Show that if $G_1, \ldots, G_n$ is a finite family of abelian groups, then the abelianization

$$(G_1 * \ldots * G_n)^{ab} \cong G_1 \oplus \ldots \oplus G_n.$$

Conclude that $\mathbb{F}_n$ is not isomorphic to $\mathbb{F}_m$ for $n \neq m$.

Exercise 6

Compute the fundamental group of the complement of a finite set in S^2.

Exercise 7

Compute the fundamental group of the complement of two intersecting lines in $\mathbb{R}^3$.

Exercise 8

Compute the fundamental group of the real projective space $\mathbb{RP}^3$. Generalize this to compute the fundamental group of $\mathbb{RP}^n$ for all n.

Exercise* 9

Let F be the group $\mathbb{Z} * \mathbb{Z}$. Show that for any $n \geq 2$ there exists a subgroup of F which is a isomorphic to $\underbrace{\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}}_{n \text{ times}}$. Hint: consider the covering $\mathbb{R}^2 \to T^2$ and remove a point from the torus.

Exercise 10

Compute the fundamental group of the following space:

$$\bigcup_{n \in \mathbb{Z}} (x - 2n)^2 + y^2 + z^2 = 1.$$

Exercise 11: the fundamental theorem of algebra

The goal of this long exercise is to prove the fundamental theorem of algebra, in the following form:

Any polynomial of the form $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $n \geq 1$ and complex coefficients has at least one complex root.

Sub-exercise

Remember that a map is said to be nullhomotopic if it is homotopic to the constant map. Show that if a function f is nullhomotopic then f_* is the zero map.

Sub-exercise

Consider the map $g\colon S^1 \subseteq \mathbb{C} \to \mathbb{C} - \{0\}$ given by $z \to z^n$. Show that g is not nullhomotopic.

Sub-exercise

Prove the theorem in the special case when

$$|a_{n-1}| + \ldots + |a_0| < 1$$

in the following way: assume that the theorem is false, and that the polynomial P has no root. Then it is possible to define $k\colon B^1 \to \mathbb{C} - \{0\}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.$$

Call h the restriction of k to S^1. Show that:

- h is nullhomotopic;
- h is homotopic to g.

Conclude that k must have at least one root.

Sub-exercise

Conclude from the previous case that any polynomial admits a root.