Exercise session 4

Algebraic Topology 2025-2026

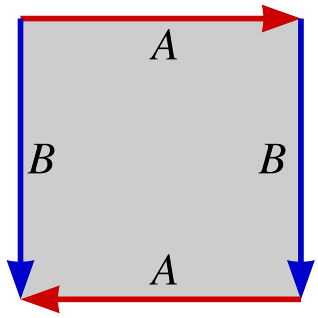
18 March, 2026

Exercise 1

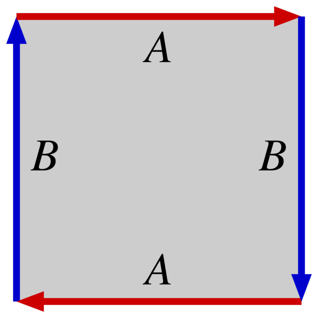
Let X, Y be path connected spaces. Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 2

Consider the Klein bottle K given by the identification square



Compute the fundamental group of the punctured Klein bottle $K - p$. What space is the identification square



homeomorphic to?

Exercise 3

Consider a map $f\colon S^n \to S^n$ such that for any x, $f(x) \neq -x$. Show that f is homotopic to the identity.

Exercise* 4

Consider a map $g\colon S^n \to S^n$ without any fixed points. Show that g^2 is homotopic to the identity.

Exercise 5

Show that each element of the free product $\mathbb{Z} * \mathbb{Z}$ can be (optional: uniquely) written as

$$a^{n_1} b^{m_1} a^{n_2} b^{m_2} \dots a^{n_k} b^{m_k}$$

with $k \geq 1$, $n_i, m_i \in \mathbb{Z}$ all nonzero except possibly n_1, m_k. Find an analogous formulation for $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Exercise 6

We'll say that a group is free if it can be written as a free product

$$*_{i \in I} \mathbb{Z}$$

for some set I. Show that any group is a quotient of a free group.

Exercise* 7

Use the previous exercise to show that for any group G, there exists a path connected topological space X_G with the property that $\pi_1(X_G) = G$. Is this space unique?