Exercise session 3

Algebraic Topology 2025-2026

11 March, 2026

Exercise 1

The following topological spaces are all homotopy equivalent to a wedge product of spheres - not necessarily of the same dimension. Find what each space is homotopy equivalent to. No formal proof is needed, a drawing is enough.

- $X_1 = \bigcup_{n\in\{-1,0,1\}}\{(x-2n)^2+y^2=1\} \subseteq \mathbb{R}^2$;
- The torus with one point removed;
- The solid torus $S^1 \times B^1$;
- $\{||x|| > 1\} \subseteq \mathbb{R}^2$
- $\{||x|| \geq 1\} \subseteq \mathbb{R}^2$
- Let $f_0, \dots f_n \colon \mathbb{R} \to \mathbb{R}^5$ be linear maps.

$$X_2 = \mathbb{R}^5 - \bigcup_i f_i([0,\infty]);$$

- The torus with n points removed;
- $S^2 \cup \{x=y=0\} \subseteq \mathbb{R}^3$;
- $\mathbb{R}^2 - \mathbb{R}_+ \times \{0\}$;

Exercise 2

Let $f \colon X \to Y, g \colon Y \to Z$ be continuous maps, and denote with

$$f_* \colon \pi_1(X,x) \to \pi_1(Y,f(x)) \text{ and } g_* \colon \pi_1(Y,f(x)) \to \pi_1(Z,gf(x))$$

the induced maps on the fundamental groups. Show that:

- $(gf)_* = g_* f_*$;
- $\mathrm{id}_{\mathrm{X}*} = \mathrm{id}_{\pi_1(X,x)}$.

Exercise 3

Let X, Y be path connected spaces. Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 4

Let $Y \subseteq X$, and denote with i the inclusion. A continuous map $r\colon X \to Y$ is said to be a retraction if $r \circ i = \mathrm{id}_Y$. Let $y \in Y$. Show that:

- r is surjective;
- $i_*\colon \pi_1(Y, y) \to \pi_1(X, y)$ is injective.

Exercise 5

Consider a map $f\colon S^n \to S^n$ such that for any x, $f(x) \neq -x$. Show that f is homotopic to the identity.

Exercise* 6

Consider a map $g\colon S^n \to S^n$ without any fixed points. Show that g^2 is homotopic to the identity.

Exercise 7

Show that a continuous map

$$\gamma\colon [0,1] \to X$$

with the property that $\gamma(0) = \gamma(1)$ is the same thing as a continuous map

$$S^1 \to X.$$

Show that if S^1 were simply connected, for every space X and point $x \in X$ one would have $\pi_1(X, x) = 0$.