Exercise session 2

Algebraic Topology 2025-2026

25 February, 2026

Exercise 1

Show (if you wish, with a drawing) that:

- There are homeomorphisms

$$\mathbb{P}^1(\mathbb{R}) \cong S^1 \text{ and } \mathbb{P}^1(\mathbb{C}) \cong S^2;$$

- The spaces $\mathbb{P}^n(\mathbb{R})$ are compact.

Exercise 2

Let $X \subseteq \mathbb{C}$ be the space

$$X = \{re^{2\pi t} \colon 0 \leq r \leq 1, t \in \mathbb{Q}\}.$$

Show that X is not a cell complex.

Exercise 3

Show that any topological space X admits an inclusion $X \hookrightarrow Y_X$ into a contractible space Y_X.

Exercise 4*

Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R} \times [0, 1]$.

Exercise 5**

Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 - \{0\}$, using only elementary means (that is using only compactness/connectedness...). Hint: use that $\mathbb{R}^2$ is the increasing union of the closed balls of radius n.

Exercise 6

The following topological spaces are all homotopy equivalent to a wedge product of spheres - not necessarily of the same dimension. Find what each space is homotopy equivalent to. No formal proof is needed, a drawing is enough.

- $X_1 = \bigcup_{n \in \{-1,0,1\}} \{(x-2n)^2 + y^2 = 1\} \subseteq \mathbb{R}^2$;
- The torus with one point removed;
- The solid torus $S^1 \times B^1$;
- $\{||x|| > 1\} \subseteq \mathbb{R}^2$
- $\{||x|| \geq 1\} \subseteq \mathbb{R}^2$
- Let $f_0, \ldots f_n \colon \mathbb{R} \to \mathbb{R}^5$ be linear maps.
$$X_2 = \mathbb{R}^5 - \bigcup_i f_i([0,\infty]);$$
- The torus with n points removed;
- $S^2 \cup \{x = y = 0\} \subseteq \mathbb{R}^3$;
- $\mathbb{R}^2 - \mathbb{R}_+ \times \{0\}$;