Exercise session 1

Algebraic Topology 2025-2026

18 February, 2026

Exercise 1

Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Exercise 2

Consider the equivalence relation $\sim$ on $\mathbb{R}^2$ generated by $(x, y) \sim (\lambda x, \lambda y)$ for $\lambda \in \mathbb{R} - \{0\}$. Prove that $\mathbb{R}^2/\sim$ is not a cell complex.

Exercise 3 [Stereographic projection]

Show that $S^n - \{p\}$, where p is any point of the sphere, is homeomorphic to $\mathbb{R}^n$. You don't have to write precise formulas, a drawing is sufficient.

Exercise 4

Show that the punctured plane $\mathbb{R}^2 - \{0\}$ is homotopy equivalent to the cylinder $S^1 \times \mathbb{R}$. Hint: show that they are both homotopy equivalent to a common space. Again, no need for a formula.

Exercise 5

Recall that if X, Y are pointed topological spaces their wedge product $X \vee Y$ is defined as the quotient of $X \coprod Y$ given by identifying the two base points.

Show that the n-punctured space $\mathbb{R}^{k+1} - \{p_1, \dots, p_n\}$ is homotopy equivalent to $\underbrace{S^k \vee \ldots \vee S^k}_{n \text{ times}}$.

Exercise 6

Show that

$$X = \{x^2 + y^2 = 1\} \cup \{y = 0\} \subseteq \mathbb{R}^2$$

Is homotopy equivalent to $S^1 \vee S^1$ (i.e. the figure eight).

Exercise 7

Recall the space $\ell^2(\mathbb{N}, \mathbb{R})$ with the topology induced by the ℓ^2 norm. Consider the infinite dimensional sphere defined as

$$S^\infty = \{(a_n) \in \ell^2(\mathbb{N}, \mathbb{R}) | \sum_n a_n^2 = 1\}.$$

Show that S^∞ is contractible.