Exercise session 8

Algebraic Topology 2024-2025

28 March, 2025

Exercise 1

Let X be the topological space given by a union of three distinct planes in $\mathbb{R}^3$. Compute the fundamental group of X for any possible configuration of planes for which X is connected. Note that the planes do not necessarily pass through the origin.

Exercise 2

Recall that we have previously constructed a homeomorphism

$$\begin{aligned} \mathbb{C}P^1 &\to S^2 \cong \mathbb{C} \cup \{\infty\} \\ [z_0, z_1] &\to z_1/z_0 \text{ if } z_0 \neq 0 \\ [z_0, z_1] &\to \infty \text{ if } z_0 = 0. \end{aligned}$$

Consider the map constructed in the following way: see S^3 as the subspace of $\mathbb{C}^2 - \{0\}$ given by $\{z_1, z_2 \in \mathbb{C}^2 | \, |z_1|^2 + |z_2|^2 = 1\}$, and consider the projection to the quotient

$$h \colon S^3 \subseteq \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1 \cong S^2.$$

The map h is called the Hopf fibration. Show that for any point $p \in S^2$, $h^{-1}(\{p\}) \cong S^1$. Show that despite this the Hopf fibration is not trivial, i.e. S^3 is not homeomorphic to the product $S^2 \times S^1$.

This is too hard to show in an exercise, but it is possible to prove that the Hopf fibration is the generator of $\pi_3(S^2) \cong \mathbb{Z}$.

Exercise 3

Compute the fundamental group of the complement of a finite set in S^2.

Exercise 4

Let F be the group $\mathbb{Z} * \mathbb{Z}$. Show, using the theory of covering spaces, that for any $n \geq 2$ there exists a subgroup of F which is a isomorphic to $\underbrace{\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}}_{n \text{ times}}$.