Exercise session 6

Algebraic Topology 2024-2025

19 March, 2025

Sub-exercise

Consider the map $g\colon S^1 \subseteq \mathbb{C} \to \mathbb{C} - \{0\}$ given by $z \to z^n$. Show that g is not nullhomotopic.

Sub-exercise

Prove the theorem in the special case when

$$|a_{n-1}| + \ldots + |a_0| < 1$$

in the following way: assume that the theorem is false, and that the polynomial P has no root. Then it is possible to define $k\colon D^2 \to \mathbb{C} - \{0\}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.$$

Call h the restriction of k to S^1. Show that h is nullhomotopic, and that h is homotopic to g. Conclude that k must have at least one root.

Sub-exercise

Conclude from the previous case that any polynomial admits a root.

Exercise 1

Recall the space $\ell^2(\mathbb{N}, \mathbb{R})$ with the topology induced by the ℓ^2 norm. Consider the infinite dimensional sphere defined as

$$S^\infty = \{(a_n) \in \ell^2(\mathbb{N}, \mathbb{R}) | \sum_n a_n^2 = 1\}.$$

Show that S^∞ is contractible.

Exercise 2

Let X be the following space:

$$\bigcup_{n \in \mathbb{Z}} (x - 2n)^2 + y^2 + z^2 = 1.$$

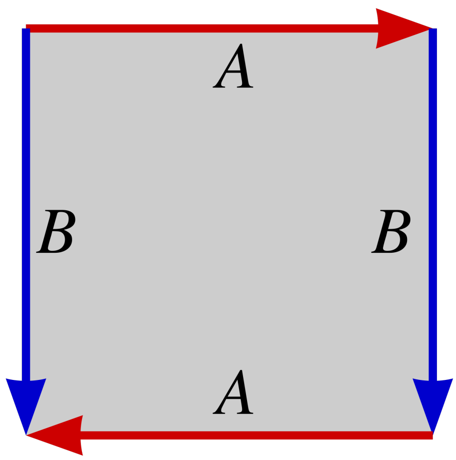
Show that X is simply connected.

Exercise 3

Calculate the fundamental group of the torus using the Seifert-Van Kampen theorem. Do not cheat by using the decomposition as a product!

Exercise 4

Recall that the Klein Bottle is defined as by identifying the sides of a square as depicted in the picture.



Compute the fundamental group of the Klein bottle minus a point.

Exercise 5

Compute the fundamental group of the Klein bottle. Hint: the previous two exercises show the way. You may not be able to give a nice description, one in terms of generators and relations suffices.

Exercise 6

Compute the fundamental group of the real projective space $\mathbb{R}P^2$. Hint: recall that $\mathbb{R}P^2$ is homeomorphic to the quotient

$$S^2/\{+1,-1\};$$

this, in turn, is homeomorphic to the quotient of D^2 where we have identified the antipodal points on its boundary S^1.