Exercise session 5

Algebraic Topology 2024-2025

12 March, 2025

Exercise 1

Let $p\colon E \to X$ be a covering map and $e \in E$. Denote $p(e)$ with x. Show that

$$p_*\colon \pi_1(E, e) \to \pi_1(X, x)$$

is injective.

Exercise 2

Consider a map $f\colon S^n \to S^n$ such that for any x, $f(x) \neq -x$. Show that f is homotopic to the identity.

Exercise 3

Consider a map $g\colon S^n \to S^n$ without any fixed points. Show that g^2 is homotopic to the identity.

Exercise 4

Show that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^2$, for $n \neq 2$.

Exercise 5: the fundamental theorem of algebra

The goal of this long exercise is to prove the fundamental theorem of algebra, in the following form:

Any polynomial of the form $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $n \geq 1$ and complex coefficients has at least one complex root.

Sub-exercise

Remember that a map is said to be nullhomotopic if it is homotopic to the constant map. Show that if a function f is nullhomotopic then $\pi_1(f)$ is the zero map.

Sub-exercise

Consider the map $g: S^1 \subseteq \mathbb{C} \to \mathbb{C} - \{0\}$ given by $z \to z^n$. Show that g is not nullhomotopic.

Sub-exercise

Prove the theorem in the special case when

$$|a_{n-1}| + \ldots + |a_0| < 1$$

in the following way: assume that the theorem is false, and that the polynomial P has no root. Then it is possible to define $k: B^1 \to \mathbb{C} - \{0\}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.$$

Call h the restriction of k to S^1. Show that:

- h is nullhomotopic;
- h is homotopic to g.

Conclude that k must have at least one root.

Sub-exercise

Conclude from the previous case that any polynomial admits a root.

Exercise 6

Recall the space $\ell^2(\mathbb{N}, \mathbb{R})$ with the topology induced by the ℓ^2 norm. Consider the infinite dimensional sphere defined as

$$S^\infty = \{(a_n) \in \ell^2(\mathbb{N}, \mathbb{R}) | \sum_n a_n^2 = 1\}.$$

Show that S^∞ is contractible.