Exercise session 3

Algebraic Topology 2024-2025

5 March, 2025

Exercise 1

The following topological spaces are all homotopy equivalent to a wedge product of spheres – not necessarily of the same dimension. Find what each space is homotopy equivalent to. No formal proof is needed, a drawing is enough.

- $\{||x|| > 1\} \subseteq \mathbb{R}^2$

- $\{||x|| \geq 1\} \subseteq \mathbb{R}^2$

- Let $f_0, \ldots f_n \colon \mathbb{R} \to \mathbb{R}^5$ be linear maps.
$$X_2 = \mathbb{R}^5 - \bigcup_i f_i([0, \infty]);$$

- The torus with n points removed;

- $S^2 \cup \{x = y = 0\} \subseteq \mathbb{R}^3$;

- $\mathbb{R}^2 - \mathbb{R}_+ \times \{0\}$;

Exercise 2

Let $f \colon S^2 \to \mathbb{R}^3$ be a continuous map. Do there always exist antipodal points $-x, x \in S^2$ such that $f(x) = f(-x)$?

Exercise 3

Let X, Y be path connected spaces. Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 4

Let $Y \subseteq X$, and denote with i the inclusion. A continuous map $r \colon X \to Y$ is said to be a retraction if $r \circ i = \text{id}_Y$. Let $y \in Y$. Show that:

- r is surjective;

- $i_* \colon \pi_1(Y, y) \to \pi_1(X, y)$ is injective.

Exercise 5

Consider a map $f \colon S^n \to S^n$ such that for any x, $f(x) \neq -x$. Show that f is homotopic to the identity.

Exercise* 6

Consider a map $g \colon S^n \to S^n$ without any fixed points. Show that g^2 is homotopic to the identity.