Exercise session 3

Algebraic Topology 2024-2025

26 February, 2025

Exercise 1

Show (if you wish, with a drawing) that:

- There are homeomorphisms
$$\mathbb{P}^1(\mathbb{R}) \cong S^1 \text{ and } \mathbb{P}^1(\mathbb{C}) \cong S^2;$$
- The spaces $\mathbb{P}^n(\mathbb{R})$ are compact.

Exercise 2*

Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 - \{0\}$, using only elementary means (that is using only compactness/connectedness...). Hint: use that $\mathbb{R}^2$ is the increasing union of the closed balls of radius n.

Exercise 3

Let X be a topological space, and I a set. Show that the canonical projection map
$$\coprod_{i \in I} X \to X$$
is a covering.

Exercise 4

The following topological spaces are all homotopy equivalent to a wedge product of spheres - not necessarily of the same dimension. Find what each space is homotopy equivalent to. No formal proof is needed, a drawing is enough.

- $X_1 = \bigcup_{n \in \{-1,0,1\}} \{(x-2n)^2 + y^2 = 1\} \subseteq \mathbb{R}^2$;
- The torus with one point removed;
- The solid torus $S^1 \times B^1$;

- $\{||x|| > 1\} \subseteq \mathbb{R}^2$
- $\{||x|| \geq 1\} \subseteq \mathbb{R}^2$
- Let $f_0, \dots f_n \colon \mathbb{R} \to \mathbb{R}^5$ be linear maps.

$$X_2 = \mathbb{R}^5 - \bigcup_i f_i([0, \infty]);$$

- The torus with n points removed;
- $S^2 \cup \{x = y = 0\} \subseteq \mathbb{R}^3$;
- $\mathbb{R}^2 - \mathbb{R}_+ \times \{0\}$;

Exercise 5

Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 6

Show that a continuous map

$$\gamma \colon [0, 1] \to X$$

with the property that $\gamma(0) = \gamma(1)$ is the same thing as a continuous map

$$S^1 \to X.$$

Show that any loop

$$\gamma \colon S^1 \to S^n$$

which is not surjective is homotopic to a constant loop. Does this imply that S^n is simply connected for $n \geq 2$?

Exercise 7

Show that if S^1 were simply connected, for every space X and point $x \in X$ one would have $\pi_1(X, x) = 0$.