Exercise session 2

Algebraic Topology 2024-2025

19 February, 2025

Recall the following fact from general topology:

Lemma 0.1. *If X, Y are topological spaces, the projection*

$$p\colon X \times Y \to Y$$

is open. If X is compact, then p is also closed.

You may prove this, but it's not an exercise.

Exercise 1

Let $\mathrm{GL}(n, \mathbb{R}) \subseteq M_{n,n}(\mathbb{R})$ be the subspace given by all the invertible matrices. Prove that $\mathrm{GL}(n, \mathbb{R})$ is a dense open subspace of $M_{n,n}(\mathbb{R})$. Is it connected?

Exercise 2

Show that the following subspaces are closed in $\mathrm{GL}(n, \mathbb{R})$:

- $\mathrm{SL}(n, \mathbb{R}) = \{M \in \mathrm{GL}(n, \mathbb{R}) | \det M = 1\}$
- $\mathrm{O}(n, \mathbb{R}) = \{M \in \mathrm{GL}(n, \mathbb{R}) | M^t M = I\}$, where I is the identity matrix. Is this connected?
- $\mathrm{SO}(n, \mathbb{R}) = \{M \in \mathrm{O}(n, \mathbb{R}) | \det M = 1\}$;

Exercise 3

Let $X \subseteq M_{n,n}(\mathbb{R})$ be the subset of the matrices A such that $A^n = 0.$[1]. Tell whether X is:

- Closed;
- Connected;
- Compact.

[1] As a consequence of the Cayley-Hamilton theorem, these are in fact all the nilpotent $n \times n$ matrices.

Exercise 4

Let $Y \subseteq M_{n,n}(\mathbb{R})$ be the subset of all matrices that have an eigenvalue $\lambda \in [0,1]$. Show that:

1. Y is closed;

2. Y is not compact.

Exercise 5

Let $X \neq \emptyset$ be a topological space. Show that its cone CX deformation retracts to a point. What happens if $X = \emptyset$?

Exercise 6

Show (if you wish, with a drawing) that:

- There are homeomorphisms

$$\mathbb{P}^1(\mathbb{R}) \cong S^1 \text{ and } \mathbb{P}^1(\mathbb{C}) \cong S^2;$$

- The spaces $\mathbb{P}^n(\mathbb{R})$ are compact.

Exercise 7*

Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 - \{0\}$, using only elementary means (that is using only compactness/connectedness...). Hint: use that $\mathbb{R}^2$ is the increasing union of the closed balls of radius n.