Exercise session 13

Algebraic Topology 2024-2025

20 May, 2025

Exercise 1

Compute the homology of the Klein bottle, both using cellular homology and the Mayer-Vietoris sequence.

Exercise 2

Recall that the torus T^2 can be defines as the product $S^1 \times S^1$. Consider the open cover of T^2 given by $U = S^1 \times (S^1 - \{p\})$ and $V = S^1 \times (S^1 - \{q\})$, where p and q are distinct points. Use this open cover and the Mayer-Vietoris sequence to compute the homology of T^2. Try to generalize this to the higher dimensional torus $T^n = (S^1)^{\times n}$.

Exercise 3: the singular homology of real projective space

In this exercise we will compute the homology of the real projective space $\mathbb{R}P^n$. Recall that by definition, $\mathbb{R}P^n$ is the quotient of the sphere S^{n+1} obtained by identifying antipodal points. Equivalently, $\mathbb{R}P^n$ can also be seen as a quotient of the disk D^n where each point of its boundary S^n is identified to its antipode. Let's first compute the homology in projective spaces of low dimensions.

Step 0

Since $\mathbb{R}P^1 \cong S^1$, this case is trivial.

Step 1

Under the point of view explained above, you can use the Mayer-Vietoris sequence to compute the homology $H_k(\mathbb{R}P^2)$. Take as U the (image in the quotient of) the whole disk minus the boundary, and as V a small neighborhood of the boundary; use this decomposition to compute $H_k(\mathbb{R}P^2)$ for all k.

Step 2

Use a similar argument to compute the homology of $\mathbb{R}P^3$.

Step 3

Generalize to the case of $\mathbb{R}P^n$.

Remark. This computation could also be carried out via cellular homology, using the cell decomposition of $\mathbb{R}P^n$.