Exercise session 12

Algebraic Topology 2024-2025

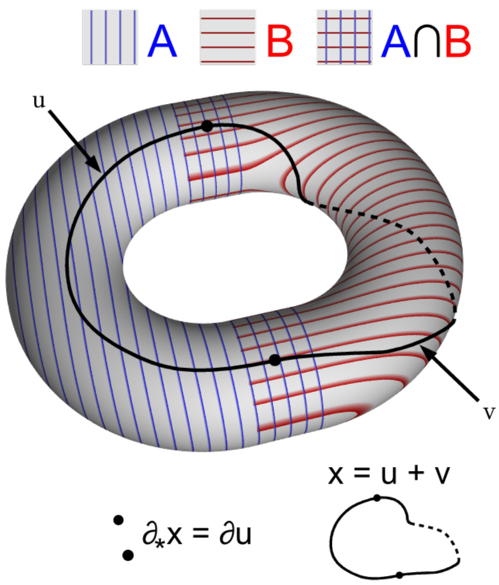
14 May, 2025

Reminder: The Mayer-Vietoris sequence

Sometimes in computation, it is important to be able to explicitly describe the boundary map appearing in the Mayer-Vietoris sequence. Let X be a topological space and A, B two open sets that cover X. Denote with i, j the inclusions $A, B \to X$ and with k, l the inclusions $A \cap B \to A, B$. Then there is a long exact sequence

$$\ldots \to H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{(k_*, l_*)} H_n(A) \oplus H_n(B) \xrightarrow{i_* - j_*} H_n(X) \to \ldots$$

The map ∂ is defined by homological means, but it is possible to give an explicit interpretation: an n-chain c in $H_{n+1}(X, R)$ can always be written (for example by barycentric subdivision) as a sum $c = u + v$ where the image of u lies in A and the image of v lies in B. Since $dc = 0$, one has $du = -dv$ and then the image of du is fully contained in the intersection $A \cap B$. Then the class $\partial[x]$ can be defined as the class of du in $H_n(A \cap B)$. Note that, despite $\partial[x]$ being defined as a boundary, it is not necessarily zero in homology because u is not an element of $C_{n+1}(A \cap B)$.



Exercise 1

Use the Mayer-Vietoris sequence to compute the homology groups of the n-dimensional sphere S^n.

Exercise 2

Compute the homology of the Klein bottle, both using cellular homology and the Mayer-Vietoris sequence.

Exercise 3

Recall that the torus T^2 can be defines as the product $S^1 \times S^1$. Consider the open cover of T^2 given by $U = S^1 \times (S^1 - \{p\})$ and $V = S^1 \times (S^1 - \{q\})$, where p and q are distinct points. Use this open cover and the Mayer-Vietoris sequence to compute the homology of T^2. Try to generalize this to the higher dimensional torus $T^n = (S^1)^{\times n}$.