Exercise session 1

Algebraic Topology 2024-2025

12 February, 2025

Exercise 1

Prove that any metrizable space is Hausdorff.

Exercise 2

Consider the equivalence relation $\sim$ on $\mathbb{R}^2$ generated by $(x, y) \sim (\lambda x, \lambda y)$ for $\lambda \in \mathbb{R} - \{0\}$. Prove that $\mathbb{R}^2/\sim$ is not Hausdorff.

Exercise 3 [Stereographic projection]

Show that $S^n - \{p\}$, where p is any point of the sphere, is homeomorphic to $\mathbb{R}^n$. You don't have to write precise formulas, a drawing is sufficient.

Exercise 4

Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Exercise 5

Show that $\mathcal{C}(S^1) \subseteq \mathcal{C}(\mathbb{R})$ consists of the functions $f : \mathbb{R} \to \mathbb{R}$ that are periodic, in the sense that $f(x+k) = f(x)$ for $\forall x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Can you characterize in the same way $\mathcal{C}(S^1 \times S^1)$?

Exercise 5

Show that the punctured plane $\mathbb{R}^2 - \{0\}$ is homotopy equivalent to the cylinder $S^1 \times \mathbb{R}$. Hint: show that they deformation retract to a common space.

Exercise 6

Recall that if X, Y are pointed topological spaces their wedge product $X \vee Y$ is defined as the quotient of $X \coprod Y$ given by identifying the two base points.

Show that the n-punctured space $\mathbb{R}^{k+1} - \{p_1, \ldots, p_n\}$ is homotopy equivalent to $\underbrace{S^k \vee \ldots \vee S^k}_{n \text{ times}}$. No formulas needed, a drawing is enough.

Exercise 7

Show that

$$X = \{x^2 + y^2 = 1\} \cup \{y = 0\} \subseteq \mathbb{R}^2$$

Is homotopy equivalent to $S^1 \vee S^1$ (i.e. the figure eight).