Exercise session 7

Algebraic Topology 2022-2023

Due after spring break

Exercise -1

Solve the last two points of exercise 10 of Homework 3.

Exercise 0

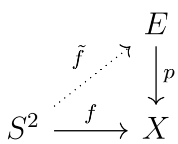
Look through the previous exercise sheets and find some exercises you would like to discuss again in class.

Exercise 1

Prove that there is a homeomorphism $\mathbb{C}P^1 \cong S^2$.

Exercise 2

Let X be a path connected topological space, $p\colon E \to X$ a covering map, and $f\colon S^2 \to X$ an arbitrary map. Show that for any $y \in S^2$ and $e \in p^{-1}\{y\}$ there exists a unique lift in the diagram



such that $\tilde{f}(y) = e$. Use this to show that any map $S^2 \to S^1$ is homotopic to the constant map. Hint: use the analogous result for $[0,1]^2$.

Exercise 3

Let X be the topological space given by a union of three distinct planes in $\mathbb{R}^3$. Compute the fundamental group of X for any possible configuration of planes for which X is connected. Note that the planes do not necessarily pass through the origin.

Exercise 4

Consider the map constructed in the following way: see S^3 as the subspace of $\mathbb{C}^2 - \{0\}$ given by $\{z_1, z_2 \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$, and consider the projection to the quotient

$$h\colon S^3 \subseteq \mathbb{C}^2 - \{0\} \to \mathbb{C}P^1 \cong S^2.$$

The map h is called the Hopf fibration. Show that for any point $p \in S^2$, $h^{-1}(\{p\}) \cong S^1$. Show that despite this the Hopf fibration is not trivial, i.e. S^3 is not homeomorphic to the product $S^2 \times S^1$.

This is too hard to show in an exercise, but it is possible to prove that the Hopf fibration is not homotopic to a constant map, showing that unlike the case shown in the previous exercise - and surprisingly! - there exist homotopically nontrivial maps $S^3 \to S^2$. In fact, one can prove that the set of pointed maps $S^3 \to S^2$ up to homotopy is the free abelian group of rank one generated by the Hopf fibration.

Exercise

Let X be the union of the unit sphere $S^2 \subseteq \mathbb{R}^3$ with the three coordinate planes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$. Calculate the fundamental group of X.

Exercise 5

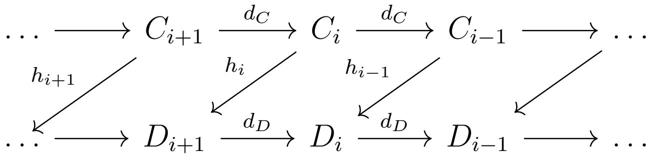
Let

$$C_\bullet = \ldots \to C_{i+1} \overset{d_C}{\to} C_i \overset{d_C}{\to} C_{i-1} \to \ldots$$

and

$$D_\bullet = \ldots \to D_{i+1} \overset{d_D}{\to} D_i \overset{d_D}{\to} D_{i-1} \to \ldots$$

be chain complexes of abelian groups, and $f, g\colon C \to D$ morphisms of chain complexes. A homotopy between f and g is by definition a collection of maps of abelian groups $\{h_i\colon C_i \to D_{i+1}\}$



such that $h_{i-1}d_C + d_D h_i = f_i - g_i$ for all i.

- Show that if there exists a homotopy between f and g, then $H_n(f) = H_n(g)\colon H_n(C) \to H_n(D)$.

- Show that if there exists a homotopy between the identity of C and the zero map, then $H_n(C) = 0$ for any n. In this case we will say that the chain complex C is contractible.
- Find an example of a chain complex with $H_n(C) = 0$ for all n which is not contractible.

Note: if you have followed or are following homological algebra this exercise is probably superfluous.

Exercise 6

Let X be the topological space obtained by rotating the circle

$$\{x = 0, (y-1)^2 + z^2 = 1\} \subseteq \mathbb{R}^3$$

around the z-axis. Note that X is homotopy equivalent to a torus with a disk in its hole. Compute the fundamental group of X.

Exercise 7*

Consider the infinite dimensional sphere defined as

$$S^\infty = \{(a_n) \in \ell^2(\mathbb{N}, \mathbb{R}) | \sum_n a_n^2 = 1\} \subseteq \ell^2(\mathbb{N}, \mathbb{R}).$$

Show that S^∞ is contractible.