Exercise session 6

Algebraic Topology 2022-2023

Due 28 March, 2023

Exercise 1

Let X be the following space:

$$\bigcup_{n\in\mathbb{Z}} (x-2n)^2 + y^2 + z^2 = 1.$$

Show that X is simply connected.

Exercise 2

Calculate the fundamental group of the punctured Klein bottle.

Exercise 3

Calculate the fundamental group of the torus using the Seifert-Van Kampen theorem. Do not cheat by using the decomposition as a product!

Exercise 4

Calculate the fundamental group of the Klein bottle. Hint: the previous two exercises show the way. You may not be able to give a nice description, one in terms of generators and relations suffices.

Exercise 5

Recall that the real projective space $\mathbb{R}P^n$ can be defined by taking the sphere S^n and identifying its antipodal points. Compute the fundamental group of $\mathbb{R}P^n$, first for $n = 1$ and then for $n \geq 1$.

Exercise 6

Calculate the fundamental group of $\mathbb{R}^3 - S^1$.

Exercise 7

Compute the fundamental groups of three copies of S^2 each tangent to the other two.

Exercise 8

Let F be the free group on two elements. Use algebraic topology to show that for any $n \geq 1$ there exists a subgroup $G \subseteq F$ which is a free group on n generators

Exercise 9

Show that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

Exercise 10

Let $p\colon E \to X$ be a covering, and suppose E arc connected. Let $Y \subseteq X$ be an arc connected subspace. Given $a \in Y$, show that if the map

$$\pi_1(Y, a) \to \pi_1(X, a)$$

induced by the inclusion is surjective, then $p^{-1}(Y)$ is arc connected.

Is the converse true assuming that $\pi_1(E) = 0$? Is the converse true in general?