Exercise session 5

Algebraic Topology 2022-2023

Due 21 March, 2023

Exercise 1: the fundamental theorem of algebra

The goal of this long exercise is to prove the fundamental theorem of algebra, in the following form:

Any polynomial of the form $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ with $n \geq 1$ and complex coefficients has at least one complex root.

Sub-exercise

Remember that a map is said to be nullhomotopic if it is homotopic to the constant map. Show that if a function f is nullhomotopic then $\pi_1(f)$ is the zero map.

Sub-exercise

Consider the map $g\colon S^1 \to \mathbb{C} - \{0\}$ given by $z \to z^n$. Show that g is not nullhomotopic.

Sub-exercise

Prove the theorem in the special case when

$$|a_{n-1}| + \ldots + |a_0| < 1$$

in the following way: assume that the theorem is false, and that the polynomial P has no root. Then it is possible to define $k\colon B^1 \to \mathbb{C} - \{0\}$ by the equation

$$k(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0.$$

Call h the restriction of k to S^1. Show that:

- h is nullhomotopic;
- h is homotopic to g.

Conclude that P must have at least one root.

Sub-exercise

Show how the special case that was just proved implies the general case.

Exercise 2

Show that any loop

$$\gamma \colon S^1 \to S^n$$

which is not surjective is homotopic to a constant loop. Does this imply that S^n is simply connected for $n \geq 2$?

Exercise 3

Show that if X is a topological space which has a point $\eta \in X$ whose closure is the whole space, then X is contractible.

Exercise 4

Show that if S^1 were simply connected for every space X and point $x \in X$ one would have $\pi_1(X, x) = 0$.

Exercise 5

Show that there does not exist a retraction of $\mathbb{R}$ to the open interval $(0, 1)$.

Exercise* 6

Let X be a path connected space. Denote with $[S^1, X]$ the set of homotopy classes of maps from S^1 to X, without any condition on the basepoint. Prove that there is a bijection between $[S^1, X]$ and the set of conjugacy classes of $\pi_1(X, x)$ for any choice of $x \in X$. Hint: use the obvious map $\pi_1(X, x) \to [S^1, X]$.