Exercise session 4

Algebraic Topology 2022-2023

Due 14 March 2023

Exercise 1

The following topological spaces are all homotopy equivalent to a wedge product of spheres - not necessarily of the same dimension. Find what each space is homotopy equivalent to.

- $X_1 = \bigcup_{n\in\{-1,0,1\}}\{(x-2n)^2+y^2=1\} \subseteq \mathbb{R}^2$;
- The torus with one point removed;
- The solid torus $S^1 \times B^1$;
- $\{||x|| > 1\} \subseteq \mathbb{R}^2$
- $\{||x|| \geq 1\} \subseteq \mathbb{R}^2$
- Let $f_0, \dots f_n \colon \mathbb{R}^5 \to \mathbb{R}^5$ be linear maps.

$$X_2 = \mathbb{R}^5 - \bigcup_i f_i([0,\infty]);$$

- The torus with n points removed;
- $S^2 \cup \{x = y = 0\} \subseteq \mathbb{R}^3$;
- $\mathbb{R}^2 - \mathbb{R}_+ \times \{0\}$;

Exercise 2

Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Exercise 3

Let **Top** be the category of topological spaces, $\mathcal{C}$ any category and

$$F \colon \mathbf{Top} \to \mathcal{C}$$

a functor which sends homotopy equivalences to isomorphisms. Show that if two continuous maps f, g are homotopic then $F(f) = F(g)$.

Exercise 4 (Exercise 3.11.1 from the notes)

In this exercise you can use that $\pi_1(S^1) = \mathbb{Z}$. Prove that any continuous map $f\colon D^2 \to D^2$ has a fixed point, i.e. a point $x \in D^2$ such that $f(x) = x$. To do this, show that a map without fixed points would induce a retraction $D^2 \to S^1$, and conclude by showing that this is impossible.

Exercise 5

Consider a map $f\colon S^n \to S^n$ such that for any x, $f(x) \neq -x$. Show that f is homotopic to the identity.

Exercise* 6

Consider a map $g\colon S^n \to S^n$ without any fixed points. Show that g^2 is homotopic to the identity.