Exercise session 3

Algebraic Topology 2022-2023

7 March, 2023

All maps are, unless otherwise stated, continuous. Let $f, g\colon X \to Y$ be two maps. We say that they are homotopic if there exists a homotopy between them, i.e. a map

$$F\colon X \times [0,1] \to Y$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all x in X. A map $f\colon X \to Y$ is said to be a homotopy equivalence if there exists a map $g\colon Y \to X$ such that gf is homotopic to the identity of X and fg is homotopic to the identity of Y. We say that two spaces are homotopy equivalent if there exists a homotopy equivalence between them.

Exercise 1

Let $p\colon E \to X$ be a covering map and $e \in E$. Denote $p(e)$ with x. Show that

$$p_*\colon \pi_1(E,e) \to \pi_1(X,x)$$

is injective.

Exercise 2

Let $Y \subseteq X$, and denote with i the inclusion. A continuous map $r\colon X \to Y$ is said to be a retraction if $r \circ i = \mathrm{id}_Y$. Let $y \in Y$. Show that:

- r is surjective;
- $i_*\colon \pi_1(Y,y) \to \pi_1(X,y)$ is injective.

Exercise 3 (Deformation retracts)

Let X be a topological space. A subspace $Y \subseteq X$ is said to be a deformation retract of X if there exists a map

$$F\colon X \times [0,1] \to X$$

said deformation retraction, such that

- $F(x, 0) = x$ for all $x \in X$;
- $F(x, 1) \in Y$ for all $x \in X$;
- $F(y, 1) = y$ for all $y \in Y$.

Show that:

- $F(-, 1)$ is a retraction; find an example of a retraction that is not a deformation retraction.
- If Y is a deformation retract of X, then X and Y are homotopy equivalent.
- S^n is a deformation retract of $\mathbb{R}^{n+1} - \{0\}$.

Exercise 4

Let X be a topological space. Show that the following conditions are equivalent:

- There exists a point $x \in X$ such that X deformation retracts to x;
- For any point $x \in X$, X deformation retracts to x;
- X is homotopy equivalent to the one-point topological space;

In this case, we will say that X is contractible. Show that any convex space is contractible. Show that if X is contractible, for any space Y, $X \times Y$ deformation retracts to Y.

Exercise 5

Show that the punctured plane $\mathbb{R}^2 - \{0\}$ is homotopy equivalent to the cylinder $S^1 \times \mathbb{R}$.

Exercise 6

Definition 1. Recall that if X, Y are pointed topological spaces their wedge product $X \vee Y$ is defined as the quotient of $X \coprod Y$ given by identifying the two base points.

Show that the n-punctured space $\mathbb{R}^{k+1} - \{p_1, \ldots, p_n\}$ is homotopy equivalent to $\underbrace{S^k \vee \ldots \vee S^k}_{n \text{ times}}$. No formulas needed, a drawing is enough.

Exercise 7

Show that

$$X = \{x^2 + y^2 = 1\} \cup \{y = 0\} \subseteq \mathbb{R}^2$$

Is homotopy equivalent to $S^1 \vee S^1$ (i.e. the figure eight).