Exercise session 2

Algebraic Topology 2022-2023

21 February, 2023

Recall the following fact from general topology:

Lemma 0.1. *If X, Y are topological spaces, the projection*

$$p: X \times Y \to Y$$

is open. If X is compact, then p is also closed.

You may prove this, but it's not an exercise.

Exercise 1

Let $X \subseteq M_{n,n}(\mathbb{R})$ be the subset of the matrices A such that $A^n = 0$.[1]. Tell whether X is:

- Closed;
- Connected;
- Compact.

Exercise 2

Let $GL(n, \mathbb{R}) \subseteq M_{n,n}(\mathbb{R})$ be the subspace given by all the invertible matrices. Prove that $GL(n, \mathbb{R})$ is a dense open subspace of $M_{n,n}(\mathbb{R})$. Is it connected?

Exercise 3 **[Convex hull]**

Let $A \subseteq \mathbb{R}^n$. Recall that its convex hull $\tilde{A}$ is defined as the union of all the segments connecting two points of A. Show that:

1. If A is compact, then $\tilde{A}$ is closed;
2. If A is open, then $\tilde{A}$ is open;
3. If A is closed, $\tilde{A}$ is not necessarily closed.

[1] As a consequence of the Cayley-Hamilton theorem, these are in fact all the nilpotent $n \times n$ matrices.

Exercise 4

Let $Y \subseteq M_{n,n}(\mathbb{R})$ be the subset of all matrices that have an eigenvalue $\lambda \in [0, 1]$. Show that:

1. Y is closed;

2. Y is not compact.

Exercise 5

Show that the following spaces are connected:

- $\mathbb{R}^2 - A$, where A is any finite subspace;

- $\mathbb{R}^2 - \mathbb{Q}^2$;

- $\mathbb{Q}^2 \cup (\mathbb{R} - \mathbb{Q})^2 \subseteq \mathbb{R}^2$.

Exercise 6*

Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 - \{0\}$, using only elementary means (that is using only compactness/connectedness... and no homology, simple connectedness...). Hint: use that $\mathbb{R}^2$ is the increasing union of the closed balls of radius n.

Exercise 7 [**Exercise 2.4.2 from the notes**]

Construct triangulations of the following topological spaces:

- The circle;

- The real line;

- The torus (use the identification of the torus as a quotient of a square);

- The plane;

- The Klein bottle.