Exercise session 1

Algebraic Topology 2022-2023

14 February, 2023

Exercise 1 [Exercise 1.1.8 from the notes]

Prove that any metrizable space is Hausdorff.

Exercise 2 [Exercise 1.2.7 from the notes]

Consider the equivalence relation $\sim$ on $\mathbb{R}^2$ generated by $(x, y) \sim (\lambda x, \lambda y)$ for $\lambda \in \mathbb{R} - \{0\}$. Prove that $\mathbb{R}^2/\sim$ is not Hausdorff.

Exercise 3 [Exercise 1.2.8 from the notes]

Consider the product topology on $\mathbb{R}^n = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ times}}$. Show that it agrees with the topology on $\mathbb{R}^n$ induced by the metric

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Exercise 4 [Box topology]

If $(X_i)_{i \in I}$ is an (in general infinite) collection of topological spaces, you have seen in class the definition of the product topology on the set $\prod_i X_i$. We can alternatively define a different topology, τ_{box}, by choosing as a base for the topology the sets of the form $\prod_i U_i$, where each U_i is an open of X_i. The topology τ_{box} is called the box topology.

- How is this topology different to the product topology?
- Show that $\left(\prod_i X_i, \tau_{\text{box}}\right)$ does not have the universal property of a product. Hint: show that the function

$$\begin{aligned} \mathbb{R} &\to \prod_{\mathbb{N}} \mathbb{R} \\ x &\to (x, x, x, \ldots) \end{aligned}$$

which is the identity on every component is not continuous in the box topology.

Exercise 5 [Stereographic projection]

Show that $S^n - \{p\}$, where p is any point of the sphere, is homeomorphic to $\mathbb{R}^n$. You don't have to write precise formulas, a drawing is sufficient.

Exercise 6

Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. To solve this exercise it is useful to know the concept of connectedness.

Exercise 7 [Exercise 1.4.2 from the notes]

Show that $\mathcal{C}(S^1) \subseteq \mathcal{C}(\mathbb{R})$ consists of the functions $f : \mathbb{R} \to \mathbb{R}$ that are periodic, in the sense that $f(x + k) = f(x)$ for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Can you characterize in the same way $\mathcal{C}(S^1 \times S^1)$?