The curvature problem, two ways

Alessandro Lehmann

24 February 2024

Universiteit Antwerpen

Infinitesimal deformations of algebras and the Hochschild cochain complex

What is an algebra?

Let R be a commutative ring. An R-algebra is composed of:

- A R-module A;
- A multiplication map

$$m \colon A \otimes_R A \to A$$

which is associative.

Infinitesimal deformations

Fix a field k of characteristic 0. Let A be a k-algebra.

Define the ring of dual numbers $k[\varepsilon]$ as the quotient k-algebra

$$k[\varepsilon] = k[x]/(x^2) = \{a + \varepsilon b |\, a, b \in k\}.$$

Define $A[\varepsilon] = A \otimes_k k[\varepsilon] = \{a + \varepsilon b | a, b \in A\}$, it is in a natural way a $k[\varepsilon]$-module.

Is it also a $k[\varepsilon]$-algebra?

Infinitesimal deformations

Definition
An infinitesimal deformation of A is the datum of a $k[\varepsilon]$-bilinear associative multiplication map

$$\star\colon A[\varepsilon] \otimes_{k[\varepsilon]} A[\varepsilon] \to A[\varepsilon]$$

which modulo ε coincides with the multiplication m of A.

Note that the reduction modulo ε is expressed by the functor $- \otimes_{k[\varepsilon]} k$, in other words the extension of scalars via the map

$$k[\varepsilon] \xrightarrow{\varepsilon=0} k.$$

Infinitesimal deformations

By $k[\varepsilon]$-linearity, $\star$ is determined by its value on A, since

$$(a + \varepsilon b) \star (c + \varepsilon d) = a \star b + \varepsilon(a \star d + b \star c).$$

We also know that $\star$ also has to reduce to the multiplication of A, so it will always be of the form

$$a \star b = ab + \varepsilon\mu(a, b)$$

for some $\mu \in \mathrm{Hom}_k(A \otimes_k A, A)$. By definition, $ab = m(a, b)$.

When does an arbitrary $\mu \in \mathrm{Hom}_k(A \otimes_k A, A)$ define in this way an infinitesimal deformation? Precisely when associated product

$$a \star b = ab + \varepsilon\mu(a, b)$$

is associative. This is an algebraic condition on μ, namely the identity

$$a\mu(b, c) - \mu(ab, c) + \mu(a, bc) - \mu(a, b)c = 0$$

for all a, b, c in A.

The Hochschild complex

Definition
Let A be as before. Its Hochschild (cochain) complex is defined as the graded k-module

$$C^n(A) = \mathrm{Hom}_k(A^{\otimes n}, A)$$

with the convention that $A^{\otimes 0} = k$. The Hochschild complex has a differential $d_H \colon C^n(A) \to C^{n+1}(A)$,

$$d_H(f)(a_0, \ldots, a_n) =$$

$$a_0 f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \ldots, a_{i-1}a_i, \ldots, a_n) - (-1)^n f(a_0, \ldots, a_{n-1}) a_n$$

for $f \in C^n(A)$. The n-th Hochschild cohomology $HH^n(A)$ is the n-th cohomology of the Hochschild complex.

The Hochschild complex and infinitesimal deformations

Back to the question of whether a certain

$$\mu \in \mathrm{Hom}_k(A \otimes_k A, A) = C^2(A)$$

defines an infinitesimal deformation. It turns out to have the following answer:

Fact: The product $a \star b = ab + \varepsilon\mu(a, b)$ on $A[\varepsilon]$ is associative if and only if $d_H\mu = 0$.

The Hochschild complex and infinitesimal deformations

Therefore, there is a bijection

$$\begin{aligned} Z^2C^\bullet(A) &\leftrightarrow \{\text{Infinitesimal deformations of } A\} \\ \mu &\leftrightarrow a \star b = ab + \varepsilon\mu(a,b). \end{aligned}$$

This descends to the quotients, and defines a bijection

$$HH^2(A) \leftrightarrow \frac{\{\text{Infinitesimal deformations of } A\}}{\text{Equivalences of deformations}}.$$

Dg-algebras and their Hochschild complex

What is a dg-algebra?

Let R be a commutative ring and M, N cochain complexes of R-modules.

Definition
The tensor product $M \otimes_R N$ is the cochain complex

$$(M \otimes_R N)^n = \bigoplus_{i+j=n} M^i \otimes_R N^j \,,\; d(m \otimes n) = d_M m \otimes n + (-1)^{|m|} m \otimes d_N n.$$

A dg R-algebra is a cochain complex A equipped with an associative multiplication chain map

$$m \colon A \otimes_R A \to A.$$

Internal Hom

Given two cochain complexes, one can also define their internal hom

$$\mathcal{H}om^i_R(M, N) = \prod_{s\in\mathbb{Z}} \mathrm{Hom}_R(M_s, N_{s+i}) = \{f_s\colon M_s \to N_{i+s}, \}$$

$$d_{\mathcal{H}} f(m) = d_N f(m) - (-1)^{|f|} f(d_M m).$$

By definition,

$$Z^0\mathcal{H}om_R(M, N) = \mathrm{Hom}_R(M, N),$$

the chain maps between M and N.

The Hochschild complex of a dg-algebra

Take a dg-algebra A over k. We define its Hochschild complex as before:

$$C(A) = \mathcal{H}om_k(A^{\otimes n}, A)$$

The Hochschild complex of a dg-algebra

Take a dg-algebra A over k. We define its Hochschild bicomplex as before:

$$C^{m,n}(A) = \mathcal{H}om_k^m(A^{\otimes n}, A)$$

This is a bigraded object; in fact, it is a bicomplex.

The Hochschild complex of a dg-algebra

$$\begin{array}{ccccccccc}
 & & \dots & & \dots & & \dots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \to & \mathcal{H}om^0(A \otimes_k A, A) & \to & \mathcal{H}om^1(A \otimes_k A, A) & \to & \mathcal{H}om^2(A \otimes_k A, A) & \to & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathcal{H}om^0(A, A) & \longrightarrow & \mathcal{H}om^1(A, A) & \longrightarrow & \mathcal{H}om^2(A, A) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\dots & \longrightarrow & \mathcal{H}om^0(k, A) & \longrightarrow & \mathcal{H}om^1(k, A) & \longrightarrow & \mathcal{H}om^2(k, A) & \longrightarrow & \dots \\
 & & \| & & \| & & \| & & \\
 & & A^0 & & A^1 & & A^2 & &
\end{array}$$

The horizontal differential d_h is the one from the internal hom, while the vertical one d_v is defined by the same formulas that we used earlier to define the Hochschild differential.

The Hochschild complex of a dg-algebra

We can take the product totalization, and obtain a cochain complex

$$C^n(A) = \prod_{i+j=n} \mathcal{H}om^i(A^{\otimes j}, A)$$

with the differential d_H given by the sum of d_h and d_v, up to a sign.

The Hochschild cohomology $HH^n(A)$ of A is the cohomology of the complex $C^\bullet(A)$.

The Hochschild complex of a dg-algebra

$$\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \to & & \mathcal{H}om^0(A \otimes_k A, A) & \to & \mathcal{H}om^1(A \otimes_k A, A) & \to & \mathcal{H}om^2(A \otimes_k A, A) & \to & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow & & \mathcal{H}om^0(A, A) & \longrightarrow & \mathcal{H}om^1(A, A) & \longrightarrow & \mathcal{H}om^2(A, A) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots \longrightarrow & & \mathcal{H}om^0(k, A) & \longrightarrow & \mathcal{H}om^1(k, A) & \longrightarrow & \mathcal{H}om^2(k, A) & \longrightarrow & \cdots \\
 & & \| & & \| & & \| & & \\
 & & A^0 & & A^1 & & A^2 & &
\end{array}$$

The Hochschild complex of a dg-algebra

Back to deformation theory: if B is an ordinary algebra, we have seen that $\mu \in Z^2C^\bullet(B)$ induces an infinitesimal deformation of B. What changes A is a dg-algebra? Here, a cocycle

$$\mu \in Z^2C^\bullet(A) \subseteq \prod_{i\geq 0} \mathcal{H}om^{2-i}(A^{\otimes i}, A)$$

is a collection $\{\mu_i\}_{i\in\{0,1,\ldots\}}$

with

$$\mu_0 \in \mathcal{H}om^2(k, A) \cong A^2,$$
$$\mu_1 \in \mathcal{H}om^1(A, A),$$
$$\mu_2 \in \mathcal{H}om^0(A \otimes_k A, A),$$
$$\dots$$

The fact that $d_H \mu = 0$ translates to some algebraic relations between the μ_i.

The Hochschild complex of a dg-algebra

How does μ deform A? In the case of an ordinary algebra B, $\mu_B \in \mathrm{Hom}_k(B \otimes B, B)$ defined the deformation

$$(B, m) \rightsquigarrow (B[\varepsilon], m + \varepsilon\mu_B).$$

For the case of a dg-algebra, suppose for simplicity $\mu_i = 0$ for $i \geq 3$;

$$(A, m, d) \rightsquigarrow (A[\varepsilon], m + \varepsilon\mu_2, d + \varepsilon\mu_1)$$

a problem: $(d + \varepsilon\mu_1)^2 \neq 0$. In fact,

$$(d + \varepsilon\mu_1)^2 = [\varepsilon\mu_0, -]_m$$

The deformed algebra is not a dg-algebra!

Curved dg-algebras

Definition
A curved dg-algebra over a commutative ring R is a graded R-module $C^\bullet$ endowed with:

- An associative degree 0 multiplication map $C \otimes_R C \to C$;
- A map $d \in \mathcal{H}om^1(C, C)$ of degree 1 which is compatible with the multiplication, called predifferential;
- An element $c \in C^2$ such that $d(c) = 0$ and $d^2 = [c, -]$, called curvature.

Note that the element c is part of the data.

The curvature problem

So in general, an Hochschild cocycle $\mu \in Z^2 C^\bullet(A)$ gives a *curved* deformation:

$$(A, m, d) \rightsquigarrow (A[\varepsilon], m + \varepsilon\mu_2, d + \varepsilon\mu_1, \varepsilon\mu_0)$$

this is the curvature problem: the Hochschild complex parametrizes not only the deformations of A as a dg-algebra, but as a more general object.

The problem here is that cdg algebras are very hard to study homologically.

Replying to your imaginary questions

Q: What about the higher μ_i?

A: An arbitrary cocycle μ actually deforms A into a $c\mathcal{A}_\infty$ algebra, the homotopical version of a cdg algebra. Indeed, there is a bijection

$$Z^2C^\bullet(A) \leftrightarrow \{c\mathcal{A}_\infty \text{deformations of } A\}.$$

For the topic of this talk, the difference between cdg and $c\mathcal{A}_\infty$ algebras is inessential: any $c\mathcal{A}_\infty$ algebra can be rectified to a cdg algebra. The difference between dg and cdg is much more important.

Replying to your imaginary questions

Q: Couldn't one just restrict to the cocycles without curvature?

A: The Hochschild complex has several intrinsic definitions, besides the explicit one that was given in this talk. Furthermore, it enjoys some invariance properties that the truncated Hochschild complex - the object that parametrizes dg deformations - does not have.

Replying to your imaginary questions

Q: Are you sure there isn't a smart way to associate to a curved cocycle a dg-deformation and find back the classical theory?

A:

Formal moduli problems and the curvature problem

Artinian algebras

We can consider deformations over more general algebras.

Definition
A local artinian k-algebra R is a commutative local k-algebra whose residue field is k and which is finite dimensional over k.

Fundamental examples are the k-algebras $k[x]/(x^n)$.

Definition
If A is a k-algebra and R an artinian k-algebra, an R-deformation of A is a flat R-algebra A_R with an isomorphism $A_R \otimes_R k \cong A$.

Note: $A_R \otimes_R k \cong A_R/\mathfrak{m}_R A_R$

Artinian algebras

To get a definition more similar to the one that we have already seen, one can note that an R-deformation of A is the same thing as an associative multiplication on the R-module $A \otimes_k R$.

Definition
A commutative dga R over k is artinian if $H^0 R$ is a local artinian k-algebra, $H^i(R) = 0$ for $i \geq 1$, and its cohomology ring is finite dimensional over k.

Denote the category of commutative artinian dg k-algebras with $\mathbf{dgart}_k$.

Warning: extreme sketchiness ahead

Definition
A (commutative) formal moduli problem is an ∞-functor

$$F\colon \mathsf{dgart}_k \to \mathcal{S}$$

to the category of spaces satisfying the following conditions:

- $F(k) \cong *$;

Formal moduli problems

- for every (homotopy) pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & \lrcorner & \downarrow \\ R_1 & \longrightarrow & R_{01} \end{array}$$

for which the maps $H^0 R_0 \to H^0 R_{01} \leftarrow H^0 R_1$ are surjective, the diagram

$$\begin{array}{ccc} F(R) & \longrightarrow & F(R_0) \\ \downarrow & \lrcorner & \downarrow \\ F(R_1) & \longrightarrow & F(R_{01}) \end{array}$$

is (homotopy) pullback.

Formal moduli problems

In general, FMP model deformation problems (deformations of schemes, sheaves...)

Theorem (Lurie, Pridham)
There is an equivalence

$$FMP_k \cong \mathbf{dgla}_k,$$

where FMP_k is the category of formal moduli problems over k and $\mathbf{dgla}_k$ is the (infinity) category of differential graded Lie algebras.

In particular, to any FMP corresponds a cochain complex.

Deformations of categories

A natural question: do deformations of dg-algebras give FMP?

First, we generalize and consider deformations of dg-categories. There is some subtlety in the shift from algebras to categories that I will not discuss.

Note also that the definition of the Hochschild complex of a dg-category is a straightforward generalization of that of a dg-algebra.

Definition
Let $\mathcal{C}$ be a dg-category over k and R an artinian dg k-algebra. A deformation of $\mathcal{C}$ over R is a dg-category $\mathcal{C}_R$ over R with an equivalence

$$\mathcal{C}_R \otimes_R k \cong \mathcal{C}.$$

Therefore we have a functor

$$\begin{aligned} \mathrm{DefCat}_{\mathcal{C}} \colon \mathbf{dgart}_k &\to \quad \mathcal{S} \\ R &\to \{R\text{-deformations of } \mathcal{C}\}. \end{aligned}$$

The functor $\mathrm{DefCat}_{\mathcal{C}}$

$\mathrm{DefCat}_{\mathcal{C}}$ is in general not a formal moduli problem! It does admit a "FMP-ification" $\mathrm{DefCat}_{\mathcal{C}}^{\wedge}$ with a universal map

$$\mathrm{DefCat}_{\mathcal{C}} \to \mathrm{DefCat}_{\mathcal{C}}^{\wedge}.$$

Theorem (Lurie)
The underlying cochain complex corresponding to $\mathrm{DefCat}_{\mathcal{C}}^{\wedge}$ is the shifted Hochschild complex $C^{\bullet}(\mathcal{C})[1]$.

The curvature problem, again

This is again the curvature problem: we have defined a functor which only sees *uncurved* deformations; by a formal argument to this we can associate a better behaved functor, which a posteriori we see being related to the Hochschild complex. The curvature problem corresponds to the fact that the natural map

$$\mathrm{DefCat}_{\mathcal{C}} \to \mathrm{DefCat}^{\wedge}_{\mathcal{C}}$$

is in general not an equivalence.

The curvature problem, again

Too see how this is related to the first part, observe that $k[\varepsilon]$ is an artinian dg-algebra. We have by definition

$$\pi_0 \mathrm{DefCat}_{\mathcal{C}}(k[\varepsilon]) = \{\text{Infinitesimal dg-deformations of } \mathcal{C} \text{ up to equivalence}\}.$$

We also have

$$\pi_0 \mathrm{DefCat}^{\wedge}_{\mathcal{C}}(k[\varepsilon]) = HH^2(\mathcal{C}).$$

Therefore the curvature problem for infinitesimal deformations is precisely the failure of the map

$$\pi_0 \mathrm{DefCat}_{\mathcal{C}}(k[\varepsilon]) \to HH^2(\mathcal{C})$$

to be bijective.

An example

It would be nice to obtain a description of $\mathrm{DefCat}^{\wedge}_{\mathcal{C}}$ in term of curved deformations, in the same way that we defined $\mathrm{DefCat}_{\mathcal{C}}$. Problem: $\mathrm{DefCat}_{\mathcal{C}}$ is defined in terms of the homotopy theory of dg-categories. It is unclear how to even define a homotopy theory of curved categories.

Living with the curvature

How to deal with this? At least two ways to go:

- Focus on situations where the curvature is not present, restricting to certain kinds of deformations or certain categories: [GLB22][LB15][BKP18]...;
- Try to develop homological algebra of curved structures: [Pos18][BD20][LD18]...

References

[BD20] Joan Bellier-Millès and Gabriel C. Drummond-Cole. *Homotopy theory of curved operads and curved algebras.* 2020.

[BKP18] Anthony Blanc, Ludmil Katzarkov, and Pranav Pandit. *Generators in formal deformations of categories.* 2018.

[GLB22] Francesco Genovese, Wendy Lowen, and Michel Van den Bergh. *T-structures on dg-categories and derived deformations.* 2022.

[LB15] Wendy Lowen and Michel Van den Bergh. *The curvature problem for formal and infinitesimal deformations.* 2015.

[LD18] Wendy Lowen and Olivier De Deken. *Filtered* $c\mathcal{A}_\infty$ *categories and functor categories.* 2018.

[Pos18] Leonid Positselski. *Weakly curved A-infinity-algebras over a topological local ring.* 2018.