Exercise Session 10 - Solutions

Algebraic Topology 2024-2025

29 April, 2025

Solution to Exercise 1. The statement is false. We wish to show that there exists a space X, with contractible subspaces $A, B \subseteq X$ with $X = A \cup B$ such that $A \cap B$ is contractible, but X itself is not contractible.

Let $X = S^1 \subseteq \mathbb{C}$ be the circle, and let A and B be an open and a closed arc respectively, whose intersection is a half-open arc. For example, define

$$A = \{e^{i\theta} \in S^1 \mid 0 < \theta < 3\pi/2\} \qquad \text{and} \qquad B = \{e^{i\theta} \in S^1 \mid \pi \leq \theta \leq 2\pi\}$$

Then clearly $A \cup B = S^1$ and we have $A \cap B = \{e^{i\theta} \in S^1 \mid \pi \leq \theta < 3\pi/2\}$. So A, B and $A \cap B$ are all contractible, but S^1 is not.

Solution to Exercise 2. The statement is true. We wish to show that there exists an exact sequence over $\mathbb{Z}$ of the form:

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xhookrightarrow{f} \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{g} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$$

Define

$$f : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} : n \mapsto (2n, n) \quad \text{and} \quad \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} : (m, n) \mapsto m + 2n$$

Note that these are well-defined $\mathbb{Z}$-linear maps. Further,

- f is injective. Indeed, take $n \in \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$ such that $f(n) = (2n, n) = (0, 0)$ in $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, that is $2n \equiv 0 \mod 8$ and $n \equiv 0 \mod 2$. This is only possible for $n = 0$ in $\mathbb{Z}/4\mathbb{Z}$.

- $\operatorname{Im} f \subseteq \operatorname{Ker} g$. It suffices to show that $g \circ f = 0$. So for $n \in \mathbb{Z}/4\mathbb{Z}$, we calculate $g(f(n)) = g(2n, n) = 2n + 2n = 4n = 0$ in $\mathbb{Z}/4\mathbb{Z}$.

- $\operatorname{Ker} g \subseteq \operatorname{Im} f$. Indeed, take $(m, n) \in \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, so $m \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $n \in \{0, 1\}$, and assume that $(m, n) \in \operatorname{Ker} g$. Thus $m + 2n \equiv 0 \mod 4$. Thus $m \equiv -2n \equiv 2n \mod 4$. This is only possible for $(m, n) = (0, 0)$, $(m, n) = (2, 1)$, $(m, n) = (4, 0)$ or $(m, n) = (6, 1)$. Note that these are precisely the images under f of 0, 1, 2 and 3 respectively.

- g is surjective. Indeed, take $n \in \mathbb{Z}/4\mathbb{Z} = \{0,1,2,3\}$. Then $g(n,0) = n$.

Solution to Exercise 3.

- Assume that $f, g : C_\bullet \to D_\bullet$ are chain maps such that there exists a *chain homotopy* $(h_n)_{n\in\mathbb{Z}}$ between f and g. That means that each h_n is an R-linear map $C_n \to D_{n+1}$ which satisfies, for all $n \in \mathbb{Z}$:

$$h_{n-1}\partial_n^C + \partial_{n+1}^D h_n = f_n - g_n$$

Then we wish to show that for any $n \in \mathbb{Z}$, $f_* = g_* : H_n(C) \to H_n(D)$.

To this end, let $[x] \in H_n(C)$ with $x \in \operatorname{Ker} \partial_n^C$. Then:

$$\begin{aligned} f_*([x]) - g_*([x]) &= [f_n(x) - g_n(x)] = [h_{n-1}\partial_n^C(x) + \partial_{n+1}^D h_n(x)] \\ &= [h_{n-1}(0)] + [0] = [0] \end{aligned}$$

where in the second equality we used the definition of a chain homotopy. In the third equality we used the fact that $x \in \operatorname{Ker} \partial_n^C$, and further that $\partial_{n+1}^D h_n(x) \in \operatorname{Im} \partial_{n+1}^D$, which is zero in the quotient $H_n(D) = \operatorname{Ker} \partial_n^D / \operatorname{Im} \partial_{n+1}^D$. Hence, we conclude that $f_*([x]) = g_*([x])$ and therefore $f_* = g_*$.

- Suppose that there is a chain homotopy between $\mathrm{id}_{C_\bullet}$ and $0 : C_\bullet \to C_\bullet$. We wish to show that $H_n(C) = 0$ for all $n \in \mathbb{Z}$.

  Note that by the first point, we have that $(\mathrm{id}_{C_\bullet})_* = 0_* : H_n(C) \to H_n(C)$ for all $n \in \mathbb{Z}$. Now $(\mathrm{id}_{C_\bullet})_* = \mathrm{id}_{H_n}(C)$ (also see the next exercise) and $0_* = 0$, the zero map. This is only possible if the R-module $H_n(C)$ is the zero module 0.

- We wish to show that there exists a chain complex $C_\bullet$ with homology groups $H_n(C) = 0$ for all $n \in \mathbb{Z}$ but which is not *contractible*. That means, there is no chain homotopy between $\mathrm{id}_{C_\bullet} : C_\bullet \to C_\bullet$ and the zero map 0.

  Recall that an exact sequence always has trivial homology groups. So consider for example the exact sequence of $\mathbb{Z}$-modules:

$$0 \longrightarrow 2\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Z} \overset{q}{\twoheadrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  with i the inclusion and q the quotient map. We interpret this as a chain complex with $2\mathbb{Z}$ in degree 2, $\mathbb{Z}$ in degree 1 and $\mathbb{Z}/2\mathbb{Z}$ in degree 0. The maps i and q are then the differentials ∂_2 and ∂_1 respectively.

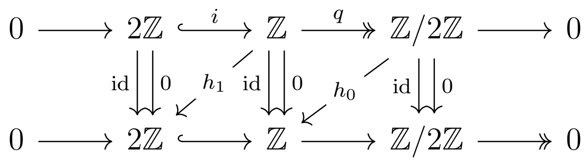
  Suppose this chain compex is contractible. Then there would exist a chain homotopy between $\mathrm{id}_{C_\bullet}$ and 0. So there would exist $\mathbb{Z}$-linear maps $h_1 : \mathbb{Z} \to 2\mathbb{Z}$ and $h_0 : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ such that

$$h_0 q + i h_1 = \mathrm{id}_\mathbb{Z} - 0$$

  But the only $\mathbb{Z}$-linear map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the zero map, so $h_0 = 0$. Hence, evaluating the above equation in 1, we find:

$$ih_1(1) = 1$$

However, the left hand side must be even, since i is the inclusion $2\mathbb{Z} \hookrightarrow \mathbb{Z}$. This is a contradiction. Hence, this chain complex is not contractible.



Solution to Exercise 4. Let $A_\bullet, B_\bullet, C_\bullet$ be chain complexes and $f : A_\bullet \to B_\bullet$ and $g : B_\bullet \to C_\bullet$ be chain maps. We wish to show that for all $n \in \mathbb{Z}$, we have

$$(g \circ f)_* = g_* \circ f_* : H_n(A) \to H_n(C) \quad \text{and} \quad (\mathrm{id}_A)_* = \mathrm{id}_{H_n(A)} : H_n(A) \to H_n(A)$$

To this end, take $[x] \in H_n(A)$ with $x \in \operatorname{Ker} \partial_n^A$. Then we have:

$$(g \circ f)_*([x]) = [(g \circ f)_n(x)] = [g_n(f_n([x]))] = g_*([f_n(x)]) = g_*(f_*([x])) = (g_* \circ f_*)([x])$$

and

$$(\mathrm{id}_{C_\bullet})_*([x]) = [\mathrm{id}_{C_n}(x)] = [x] = \mathrm{id}_{H_n(C)}([x])$$

which proves the statement.

Solution to Exercise 5. Let $f : A_\bullet \to B_\bullet$ be a *chain homotopy equivalence*. That means that there exists a chain map $g : B_\bullet \to A_\bullet$ and chain homotopies between $g \circ f$ and $\mathrm{id}_{A_\bullet}$, and between $f \circ g$ and $\mathrm{id}_{B_\bullet}$. We wish to show that f is a *quasi-isomorphism*. That means that $f_* : H_n(A) \to H_n(B)$ is an isomorphism for all $n \in \mathbb{Z}$.

Using the first point of Exercise 3, we find that $(g \circ f)_* = (\mathrm{id}_{A_\bullet})_*$ and $(g \circ f)_* = (\mathrm{id}_{B_\bullet})_*$ for all $n \in \mathbb{Z}$. Using Exercise 4, it follows that $g_* \circ f_* = \mathrm{id}_{H_n(A)}$ and $f_* \circ g_* = \mathrm{id}_{H_n(B)}$. Hence, f_* is an isomorphism with inverse given by g_*.

Further, we wish to find an example of quasi-isomorphism which is not a chain homotopy equiavelence. For this, we can reuse our example from third point of Exercise 3. Let $C_\bullet$ denote the chain complex

$$0 \longrightarrow 2\mathbb{Z} \xhookrightarrow{\ i\ } \mathbb{Z} \xrightarrow{\ q\ } \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

and let 0 denote the chain complex which is everywhere 0. Then let $f : C_\bullet \to 0$ be the unique chain map (se $f_n : C_n \to 0$ is the zero map). Since all the homology groups of both $C_\bullet$ and 0 are trivial, f is a quasi-isomorphism. However, suppose f is a chain homotopy equivalence. Then we would have a chain homotopy between $0 = 0 \circ f$ and $\mathrm{id}_{C_\bullet}$, which is impossible by what we have shown in Exercise 3.

Remark. In fact, for a general chain complex $C_\bullet$ we have that $C_\bullet$ is contractible if and only if the unique chain map $C_\bullet \to 0$ is a chain homotopy equivalence. Can you see why?

Note that this is completely analogous to topological spaces: A space X is contractible if and only if the unique continuous map to a point $X \to \{*\}$ is a homotopy equivalence.

Solution to Exercise 6. Let $f : A_\bullet \to B_\bullet$ and $g : B_\bullet \to C_\bullet$ be chain maps. We wish to show that if two out of f, g and gf are quasi-isomorphisms, then so is the third.

Consider the induced morphisms on homology $f_* : H_n(A) \to H_n(B)$ and $g_* : H_n(B) \to H_n(C)$ for every $n \in \mathbb{Z}$. Note that by Exercise 4, we have $(gf)_* = g_* \circ f_*$. Now suppose for example that gf and f are quasi-isomorphisms. By definition, we have that $g_* \circ f_*$ and f_* are isomorphisms for every $n \in \mathbb{Z}$. Hence, also $g_* = (g_* \circ f_*) \circ f_*^{-1}$ is an isomorphism for every $n \in \mathbb{Z}$ and thus g is a quasi-isomorphism as well. A similar argument shows the other cases.