Exercise session 9

Algebraic Topology 2024-2025

2 April, 2025

Exercise 1

Compute the fundamental group of the complement of a finite set in S^2.

Exercise 2

Compute the fundamental group of the complement of a two intersecting lines in $\mathbb{R}^4$.

Exercise 3

Let F be the group $\mathbb{Z} * \mathbb{Z}$. Show, using the theory of covering spaces, that for any $n \geq 2$ there exists a subgroup of F which is a isomorphic to $\underbrace{\mathbb{Z} * \mathbb{Z} * \ldots * \mathbb{Z}}_{n \text{ times}}$.

Exercise 4

Let (X, x_0) be a pointed topological space. Define then the set ΩX as the set of all loops $\gamma\colon [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$. One can make this into a topological space by using the compact-open topology; it is naturally pointed by taking $\gamma_0 = c_{x_0}$ the constant path at x_0. Show that there is an isomorphism

$$\pi_i(\Omega X, c_{x_0}) \cong \pi_{i+1}(X, x_0),\ i \geq 0.$$

Exercise 5

Let X be as above. Recall the construction of the *reduced suspension*

$$\Sigma X = \frac{X \times [0,1]}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times [0,1]}.$$

Show that ΣX is a pointed space. Convince yourself that $\Sigma S^n \cong S^{n+1}$. Show that any map $X \xrightarrow{f} Y$ induces a map $\Sigma X \xrightarrow{\Sigma f} \Sigma Y$. Use this to construct the Freudenthal homomorphism

$$\pi_i(X, x_0) \to \pi_{i+1}(\Sigma X, \Sigma x_0).$$

Note that this, despite not always being an isomorphism, is defined for all i.

Exercise 6

Show that for any topological spaces X, Y, there is a bijection between basepoint-preserving maps $\Sigma X \to Y$ and basepoint-preserving maps $X \to \Omega Y$. One can show that the topology on ΩX makes it so that continuous maps correspond to continuous maps. Show that this bijection defines two maps

$$X \to \Omega\Sigma X \text{ and } \Sigma\Omega X \to X$$

for any pointed topological space X.