Exercise session 11

Algebraic Topology 2024-2025

6 May, 2025

Exercise 1

Let $A_\bullet, B_\bullet$ and $C_\bullet$ be chain complexes of abelian groups. Let

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

be a short exact sequence of complexes; by definition, this means that for any n, the induced sequence

$$0 \to A_n \to B_n \to C_n \to 0$$

is an exact sequence of abelian groups. The goal of this exercise is to show the existence of a long exact sequence

$$\ldots \to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} H_{n-1}(B) \to \ldots$$

This is done in steps:

1. Show that the sequence

   $$H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C)$$

   is exact at $H_n(B)$, i.e. $\operatorname{Ker} g_* = \operatorname{Im} f_*$.

2. Find a procedure to construct a map

   $$H_n(C) \xrightarrow{\partial} H_{n-1}(A).$$

3. Show that the map ∂ is well-defined. Is it induced by a (degree -1) chain map $C_\bullet \to A_\bullet$?

4. Show that the sequence

   $$\ldots \to H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f_*} H_{n-1}(B) \to \ldots$$

   is exact at $H_n(A)$, i.e. $\operatorname{Ker} f_* = \operatorname{Im} \partial$.

5. Show exactness at $H_n(C)$, i.e. that $\operatorname{Ker} \partial = \operatorname{Im} g_*$.

Exercise 2

Let

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

be a short exact sequence of complexes. how that if two out of $A_\bullet, B_\bullet, C_\bullet$ are acyclic (that is, their homology vanishes) so is the third. Show that $C_\bullet$ (resp. $A_\bullet$) is acyclic if and only if f (resp. g) is a quasi-isomorphism (recall that f is said to be a quasi-isomorphism if and only if f_* is an isomorphism).

Exercise 3

Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of vector spaces. Show that

$$\dim A - \dim B + \dim C = 0.$$

Exercise 4 (Optional)

Let

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

and

$$0 \to A'_\bullet \to B'_\bullet \to C'_\bullet \to 0$$

be short exact sequences of complexes, and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \longrightarrow & B_\bullet & \longrightarrow & C_\bullet & \longrightarrow & 0 \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow h & & \downarrow \\ 0 & \longrightarrow & A'_\bullet & \longrightarrow & B'_\bullet & \longrightarrow & C'_\bullet & \longrightarrow & 0 \end{array}$$

be a commutative diagram with f, g, h chain maps. Show that if two out if f, g, h are quasi-isomorphisms, so is the third.