Exercise session 10

Algebraic Topology 2024-2025

29 April, 2025

Exercise 1

Prove or find a counterexample to the following statement: let X be a topological space which is the union of two contractible subspaces $A, B \subseteq X$ with contractible intersection. Then X is contractible.

Exercise 2

Determine whether one can have an exact sequence of the form

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0.$$

Exercise 3

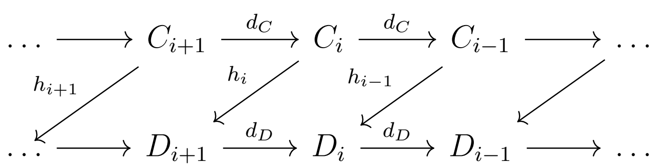
Let

$$C_\bullet = \ldots \to C_{i+1} \overset{d_C}{\to} C_i \overset{d_C}{\to} C_{i-1} \to \ldots$$

and

$$D_\bullet = \ldots \to D_{i+1} \overset{d_D}{\to} D_i \overset{d_D}{\to} D_{i-1} \to \ldots$$

be chain complexes of abelian groups, and $f, g\colon C_\bullet \to D_\bullet$ morphisms of chain complexes. A homotopy between f and g is by definition a collection of maps of abelian groups $\{h_i\colon C_i \to D_{i+1}\}$



such that $h_{i-1}d_C + d_D h_i = f_i - g_i$ for all i.

- Show that if there exists a homotopy between f and g, then $H_n(f) = H_n(g)\colon H_n(C) \to H_n(D)$.
- Show that if there exists a homotopy between the identity of C and the zero map, then $H_n(C) = 0$ for any n. In this case we will say that the chain complex $C_\bullet$ is contractible.

- Find an example of a chain complex with $H_n(C) = 0$ for all n which is not contractible.

Exercise 4

Let $A_\bullet, B_\bullet, C_\bullet$ be chain complexes of abelian groups and let

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

be chain maps. Show that $(\mathrm{id}_A)_* = \mathrm{id}_{H_\bullet(A)}$ and $(g \circ f)_* = g_* \circ f_*$.

Exercise 5

Let $A_\bullet \xrightarrow{f} B_\bullet$ be a chain map. The map f is said to be a quasi-isomorphism if $f_* \colon H_\bullet(A) \to H_\bullet(B)$ is an isomorphism; f is said to be a homotopy equivalence if there exists chain map $B_\bullet \xrightarrow{g} A_\bullet$ such that fg is homotopic to id_B and gf is homotopic to id_A. Show that any homotopy equivalence is a quasi-isomorphism. Find an example of a quasi-isomorphism that is not a homotopy equivalence.

Exercise 6

Let $A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$ be chain maps. Show that if two out of f, g, fg are quasi-isomorphisms, so is the third.