# Homework 4

Algebraic Topology 2024-2025

Due 7 may, 2025

**Exercise 1.** Consider the following two topological spaces:

- $X = \Delta^2/\partial\Delta^2$ where $\Delta^2$ is the topological 2-simplex and $\partial\Delta^2$ is the union of its three edges (its boundary).
- $Y = \Delta^2 \amalg_i \Delta^2$, the space $\Delta^2$ with $\Delta^2$ attached along the inclusion $i : \partial\Delta^2 \hookrightarrow \Delta^2$ (see p.13 in Hatcher). We may interpret this space as two triangles which are glued to each other along their boundary.

Show that both $X$ and $Y$ are homeomorphic to the sphere $S^2$. (*Hint:* Use the fact that $\Delta^2 \cong D^2$ and $\partial\Delta^2 \cong S^1$.)

**Exercise 2.** Note that $\Delta^2$ is cell complex with three 0-cells (the vertices), three 1-cells (the edges without the vertices) and one 2-cell (the interior of the triangle). This induces cell structures on $X$ and $Y$, which correspond to the two familiar cell structures on $S^2$. Convince yourself that this is true (you do not need to prove this).

- Just like the example we saw in class, construct a chain complex $C_\bullet^{CW}(X)$ such that for all $n \in \mathbb{N}$, $C_n^{CW}(X)$ is the free abelian group with basis given by the $n$-cells of $X$. What would be a reasonable definition of the differential? (*Hint:* Get inspiration from the definition of the singular chain complex.)
- Similarly, construct a chain complex $C_\bullet^{CW}(Y)$ with the same property for $Y$. Again find a reasonable definition for the differential.

**Exercise 3.** Show that $C_\bullet^{CW}(X)$ and $C_\bullet^{CW}(Y)$ have isomorphic homology groups.