Exercise session 9

Algebraic Topology 2022-2023

Due 2 May

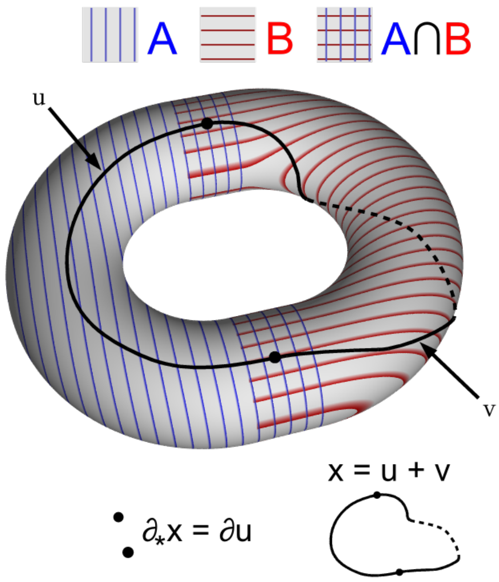
In the following I will often write $H_n(X)$ instead of $H_n(X, R)$, for brevity. Always assume an arbitrary base ring R.

Reminder: The Mayer-Vietoris sequence

Since in class I went through it very quickly, I will write something about the Mayer-Vietoris sequence that is useful for the exercises. Let X be a topological space and A, B two open sets that cover X. Denote with i, j the inclusions $A, B \to X$ and with k, l the inclusions $A \cap B \to A, B$. Then there is a long exact sequence

$$\ldots \to H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B,) \xrightarrow{(k_*, l_*)} H_n(A) \oplus H_n(B) \xrightarrow{i_* - j_*} H_n(X) \to \ldots$$

The map ∂ is defined by homological means, but it is possible to give an explicit interpretation: an n-chain c in $H_{n+1}(X, R)$ can always be written (for example by baricentric subdivision) as a sum $c = u + v$ where the image of u lies in A and the image of v lies in B. Since $dc = 0$, one has $du = -dv$ and then the image of du is fully contained in the intersection $A \cap B$. Then the class $\partial[x]$ can be defined as the class of du in $H_n(A \cap B, R)$. Note that, despite $\partial[x]$ being defined as a boundary, it is not necessarily zero in homology because u is not an element of $C_{n+1}(A \cap B, R)$.



Application: the homology of spheres

Let's use the Mayer-Vietoris sequence to compute the homology of spheres. This is also in the notes, but I am writing it here as a guide for the exercises. First, the case of S^1. Cover S^1 with two opens U, V which are both contractible and such that the intersection is homotopy equivalent to two points (note that there is no connectivity hypothesis in the Mayer-Vietoris sequence!). S^1 is arc connected, so $H_0(S^1, R) = R$. It's easy to show that all the homology groups for $n \geq 2$ vanish: we have an exact sequence

$$0 = H^n(U) \oplus H^n(V) \to H^n(S^1) \to H^n(U \cap V) = 0$$

so $H^n(S^1) = 0$ for $n \geq 2$. The case $n = 1$ is slightly more complicated, because in this case the exact sequence is

$$0 \to H^1(S^1) \to H^0(U \cap V) \to H^0(U) \oplus H^0(V) \to H^0(X) \to 0$$

which concretely is an exact sequence

$$0 \to H^1(S^1) \to R \oplus R \xrightarrow{f} R \oplus R \to R \to 0$$

so that $H^1(S^1)$ is isomorphic to the kernel of f. Unraveling the definitions, one finds that $f(x, y) = (x+y, x+y)$ (check this!) and therefore $H_1(S^1) = R$.

This generalizes easily to higher dimensions: consider the sphere S^n for $n \geq 2$, and cover it by two opens U, V which are contractible and whose intersection is homotopy equivalent to the sphere S^{n-1} (for example, you can take U and V to be the whole sphere minus the north and south pole respectively). Then the Mayer-Vietoris sequence gives, for $k \geq 1$, an exact sequence

$$0 \to H_k(S^n) \to H_{k-1}(S_{n-1}) \to 0$$

so $H_k(S^n) \cong H_{k-1}(S^{n-1})$. Therefore by induction

$$H_k(S^n, R) = \begin{cases} R & \text{for } k = 0 \text{ and } k = n; \\ 0 & \text{otherwise.} \end{cases}$$

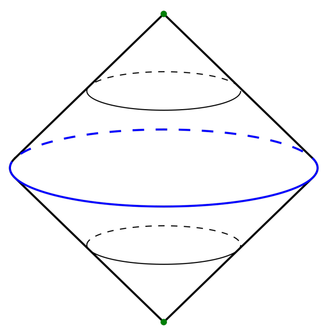
Exercise 1

Compute the homology of the following spaces;

- The wedge sum $S^n \vee S^m$;
- $\mathbb{R}^n - \mathbb{R}^m$, for $n > m \geq 0$;
- The bouquet $\underbrace{S^1 \vee \ldots \vee S^1}_{n \text{ times}}$;
- $\mathbb{R}^3 - S^1$.

Exercise 2

Let X be a topological space. Consider the cylinder $X \times [0, 1]$, and define the suspension SX as the space obtained by $X \times [0, 1]$ by collapsing to a point the two faces $X \times \{0\}$ and $X \times \{1\}$ (each face to a different point, not the same one). One can think of SX as the space constructed by stretching X to a cylinder and then pinching the end points; as an example, the suspension of S^1 is S^2, and in general the suspension of S^n is S^{n+1}.



Compute the homology of SX in terms of the homology of X.

Exercise 3

Let X_n be the product $\underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$. Show that

$$H_k(X_n, R) = \begin{cases} R^{\binom{n}{k}} & \text{for } k \leq n \\ 0 & \text{for } k > n. \end{cases}$$

You may want to use induction and the identity $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$.

Exercise 4

Show that, to to a decomposition of a space X into the union of its path connected components $X = \cup_\alpha X_\alpha$ corresponds a decomposition of its singular homology

$$H_n(X, R) \cong \bigoplus_\alpha H_n(X_\alpha, R).$$

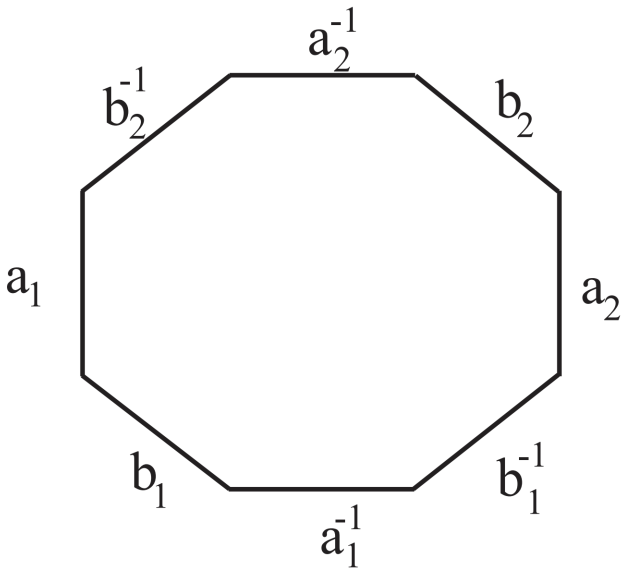
Exercise 5

Recall the characterization of the torus T^1 as the quotient of a square obtained by identifying the opposite sides.

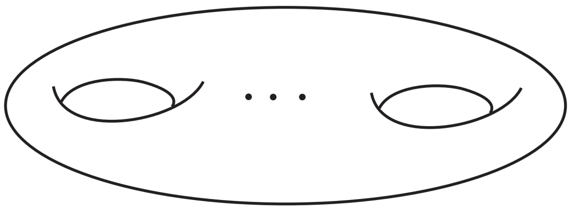
- Consider a regular polygon with $4g$ sides; denote its sides, ordered in a circular way,

  $$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, \ldots, a_g^{-1}, b_g^{-1}.$$

  Orientate the boundary by taking the sides denoted with a_i, b_i with the same orientation as the boundary and the ones denoted with a_i^{-1}, b_i^{-1} with the opposite one. Below is the case $g = 2$.



Denote with M_g the surface obtained by identifying a_i with a_i^{-1} and b_i with b_i^{-1}. This is called the genus g surface, or torus with g holes.



- Calculate $\pi_1(M_g)$ (as usual, by generators and relations).
- Compute the homology of M_g.

Exercise* 6

The goal of this exercise is to determine the relation between the first homology group and the fundamental group. Let X be a path connected topological space, and $x \in X$ any point.

Begin by constructing a map $a\colon \pi_1(X, x) \to H_1(X, \mathbb{Z})$ in the following way: any loop $\gamma\colon [0,1] \to X$ defines tautologically a singular 1-chain obtained by identifying $[0,1]$ with $|\Delta_1|$; call this chain $a(\gamma)$.

- Show that $a(\gamma)$ is a cycle, so that it defines an element in $H_1(X, \mathbb{Z})$;

- Show that a is well defined, i.e. that it only depends on the (based) homotopy class of γ;
- Show that a is surjective;
- Prove that the commutator subgroup $[\pi_1(X,x), \pi_1(X,x)] \subseteq \pi_1(X,x)$ lies in the kernel of a.
- Prove that any element in the kernel of a lies in the commutator subgroup;
- Conclude that there is an isomorphism

  $$H_1(X,\mathbb{Z}) \cong \frac{\pi_1(X,x)}{[\pi_1(X,x), \pi_1(X,x)]}.$$

  This quotient is known as the abelianization of $\pi_1(X,x)$.