Exercise session 10

Algebraic Topology 2022-2023

Due 9 May

Exercise 1: the singular homology of real projective space

In this exercise we will compute the homology of the real projective space $\mathbb{R}P^n$ with integer coefficients. The case with arbitrary coefficients is analogous, but let's keep integers coefficients for simplicity.

Recall that by definition, $\mathbb{R}P^n$ is the quotient of the sphere S^{n+1} obtained by identifying antipodal points.

As a first step, observe that $\mathbb{R}P^n$ can also be seen as a quotient of the disk D^n: namely, you can see the disk as the upper emisphere of the sphere; under this point of view, $\mathbb{R}P^n$ is a the quotient of D^n where each point of its boundary S^n is identified to its antipode. Let's first compute the homology in projective spaces of low dimensions.

Step 0

Since $\mathbb{R}P^1 \cong S^1$, this case is trivial.

Step 1

Under the point of view explained above, you can use the Mayer-Vietoris sequence to compute the homology $H_k(\mathbb{R}P^2, \mathbb{Z})$. Take as U the (image in the quotient of) the whole disk minus the boundary, and as V a small neighborhood of the boundary; use this decomposition to compute $H_k(\mathbb{R}P^2, \mathbb{Z})$ for all k.

Step 2

Use a similar argument to compute the homology of $\mathbb{R}P^3$.

Step 3

Generalize to the case of $\mathbb{R}P^n$.

Exercise 2

Show that if $Y \subseteq X$ is a retract, then the induced map $H_n(Y, R) \to H_n(X, R)$ is injective. As an application, show that there cannot exist a retraction $D^n \to S^{n-1}$ of a ball onto its boundary.

Exercise 3

Show that any continuous map $D^n \to D^n$ must have a fixed point.